Solution Set 2

Math 23a
October 8, 2002

7. We recall that a sequence $\{a_n\}$ is a Cauchy sequence provided that, given any $\epsilon > 0$, there exists a natural number N such that, for all integers $m, n > N$, we have $|a_n - a_m| < \epsilon$. We would like to show that the sequence $a_n = n^{-2}$ is Cauchy.

Fix $\epsilon > 0$. Find a natural number N such that $N^2 > 2/\epsilon$, or equivalently, $2/N^2 < \epsilon$ (we can do this by the unboundedness of the natural numbers; cf. the next problem). Pick $n, m > N$. We have

$$\left|\frac{1}{n^2} - \frac{1}{m^2}\right| \leq \left|\frac{1}{n^2}\right| + \left|\frac{1}{m^2}\right| < \frac{1}{N^2} + \frac{1}{N^2} = \frac{2}{N^2} < \epsilon$$

by the triangle inequality and the fact that $a > b \implies 1/a < 1/b$. Thus $\{1/n^2\}$ is a Cauchy sequence.

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8. Suppose that there were an upper bound for $\mathbf{N} \subset \mathbf{R}$. By the completeness property/axiom of the real numbers, then, there exists a least upper bound b, i.e. a number $b \in \mathbf{R}$ such that $n \leq b$ for all $n \in \mathbf{N}$. So there must be a natural number n such that $n > b - 1/2$ (otherwise $b - 1/2 < b$ would be an upper bound for $\mathbf{N}$ that is smaller than b, a contradiction). By Peano's axioms, $n + 1 \in \mathbf{N}$, but

$$n > b - 1/2 \implies n + 1 > b + 1/2 > b$$

which is a contradiction. Thus such a b could not have existed, so $\mathbf{N}$ must be unbounded above.

Notes on these problems:

(1) The supremum of a subset S of the real numbers that is bounded above is *not* necessarly contained in S. For instance, the supremum of $(0, 1)$ is 1, which is not contained in $(0, 1)$. Therefore, in problem 8 you were not allowed to assume that the least upper bound of $\mathbf{N}$ is a natural number.

(2) In order to prove that a sequence $\{a_n\}$ is Cauchy, *given* an ϵ you have to *find* an N such that $|a_n - a_m| < \epsilon$ for *all* $m, n > N$. It is not enough to show that $|a_n - a_{n+1}| \to 0$. For instance, if we define $a_1 = 1$ and $a_n = a_{n-1} + 1/n$ for $n > 1$ then certainly $|a_n - a_{n+1}| \to 0$, but $a_n \to \infty$ as $n \to \infty$ and are thus definitely not a Cauchy sequence (since all Cauchy sequences converge in the real numbers, by completeness).

(3) It is generally true that convergent sequences are Cauchy. However, if you want to use this fact, you have to prove it, or at least cite it, since it is not immediately obvious.

(4) As a general strategy for problems like problem 7, where you have to find *some* δ or N for a given ϵ, you can *way* overestimate if you want. In pure math, it doesn't matter if your bound is very good, as long as it works. For instance, in my solution to problem 7, if I'd wanted to work a bit harder I could have gotten away with choosing an N such that $\epsilon < 1/N^2$ instead of $2/N^2$, but I didn't want to work a bit harder, so I didn't, and my proof still works fine. Plus it's shorter.

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