

Homework 3: Part A - Problem 2

a. Observe that:

$$\begin{aligned}
 \nabla(f+g) &= (D_1(f+g), D_2(f+g), \dots, D_n(f+g)) \\
 &\quad \text{(by the definition of the gradient)} \\
 &= (D_1(f) + D_1(g), D_2(f) + D_2(g), \dots, D_n(f) + D_n(g)) \\
 &\quad \text{(by linearity of differentiation operators)} \\
 &= (D_1(f), D_2(f), \dots, D_n(f)) + (D_1(g), D_2(g), \dots, D_n(g)) \\
 &\quad \text{(by standard vector addition)} \\
 &= \nabla(f) + \nabla(g) \text{ (by definition of the gradient).}
 \end{aligned}$$

b. Now observe that

$$\begin{aligned}
 \nabla(fg) &= (D_1(fg), D_2(fg), \dots, D_n(fg)) \\
 &= (fD_1(g) + D_1(f)g, fD_2(g) + D_2(f)g, \dots, fD_n(g) + D_n(f)g) \\
 &\quad \text{(by the product rule for univariate differentiation)} \\
 &= (fD_1(g), fD_2(g), \dots, fD_n(g)) + (D_1(f)g, D_2(f)g, \dots, D_n(f)g) \\
 &\quad \text{(by standard vector addition)} \\
 &= (\nabla f)g + f(\nabla g) \text{ (by definition of gradient).}
 \end{aligned}$$

c. This shall be proven by induction on m . Clearly, the formula holds for $m = 1$ (as it reduces to $\nabla(f) = \nabla(f)$). Now suppose n is a positive integer such that the formula holds for $m = n$, i.e. $\nabla(f^n) = nf^{n-1}\nabla(f)$. Now, $f^{n+1} = f^n * f$; therefore, by part b) (which we can use because f^n , being a product of differentiable functions, is itself differentiable),

$$\begin{aligned}
 \nabla(f^{n+1}) &= \nabla(f^n * f) = \nabla(f^n) * f + f^n * \nabla(f) \\
 &= nf^{n-1}\nabla(f) * f + f^n * \nabla(f) \\
 &= nf^n\nabla(f) + f^n\nabla(f) \\
 &= (n+1)f^n\nabla(f) = (n+1)f^{(n+1)-1}\nabla(f)
 \end{aligned}$$

which implies that the formula holds for $m = n+1$ as well. Therefore, the formula holds for each positive integer n by the process of mathematical induction.

d. When $g(\mathbf{x}) \neq 0$, $\nabla(f/g) = (D_1(f/g), \dots, D_n(f/g))$ (by definition of gradient)

$$\begin{aligned}
 &= ((D_1(f)g - fD_1(g))/g^2, \dots, (D_n(f)g - fD_n(g))/g^2) \\
 &\quad \text{(by the quotient rule in univariate calculus)}
 \end{aligned}$$

$$\begin{aligned}
&= (D_1(f)/g, \dots, D_n(f)/g) - ((fD_1(g))/g^2, \dots, (fD_n(g))/g^2) \\
&\quad \text{(by standard vector addition)} \\
&= 1/g(D_1(f), \dots, D_n(f)) - f/g^2(D_1(g), \dots, D_n(g)) \\
&\quad \text{(by standard scalar multiplication)} \\
&= 1/g\nabla(f) - f/g^2\nabla(g) \text{ (by definition of gradient);} \\
&\text{this can also be written out as} \\
&1/g^2(\nabla(f)g - f\nabla(g)).
\end{aligned}$$

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