If you don't understand anything about any of the solutions here, feel free to e-mail `zeyligerfas.harvard.edu`. Homework problems have a tendency to creep up on exams, so be sure you know how to do all the assigned homework.

4

Recall

$$V = (\mathbb{Z}/n\mathbb{Z})^n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in \{0, 1, \ldots, p-1\}\}$$

a. (3 points) We wish to determine the number of vectors in V. There are p choices for every coordinate a_i, and there are n coordinates, so there are p^n vectors in V.

b. (4 points) We wish to determine the number of one-dimensional subspaces of V. By definition, every one-dimensional subspace W is spanned by one basis vector. This basis vector could be any non-zero vector in V ($\vec{0}$ spans a 0-dimensional vector space). There are $p^n - 1$ non-zero vectors in W. Now, we note that any 1-dimensional vector space $W = \text{span}\{v\}$ has $p-1$ potential bases: $\{v\}, \{2v\}, \ldots, \{(p-1)v\}$. So we need to divide by $p-1$, to conclude there are

$$\frac{p^n - 1}{p - 1}$$

one-dimensional subspaces.

c. (3 points) Now we wish to determine the number of two-dimensional subspaces. A two-dimensional vector subspace is determined by the two linearly independent vectors that make up its basis. We can choose any of the $p^n - 1$ non-zero vectors as the first vector v. There are $p^n - p$ vectors that are linearly independent from v. Note that order doesn't matter, so we divide by 2! (I took off a point if you didn't mention this), so there are

$$\frac{(p^n - 1)(p^n - p)}{2}$$

unordered pairs of linearly independent vectors in V.

Now, consider a two-dimensional subspace W. There are p^2 vectors in W. And W is spanned by any pair of linearly independent vectors in W. There are

$$\frac{(p^2 - 1)(p^2 - p)}{2}$$

such pairs. We may now conclude that there are

$$\frac{(p^n - 1)(p^n - p)}{(p^2 - 1)(p^2 - p)}$$

2-dimensional vector subspaces of V.

Note: The space $W = \text{span}\{(1,1)\} = \{(x,x) \mid x \in \mathbb{R}\}$ is a 1-dimensional subspace of $\mathbb{R}^n$. It is not two-dimensional. The space $(\mathbb{Z}/n\mathbb{Z})^n$ is n-dimensional. It is spanned, for example, by $\{(1,0,0,\ldots,0), (0,1,0,\ldots,0), \ldots, (0,0,\ldots,0,1)\}$. Be sure you understand the distinction between n-tuples and n-dimensional spaces before the midterm; come to any CA's office hours if you have questions!

5

$$\begin{aligned} V(c) &= \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous}, f(0) = c\} \\ &= \{f \in C[0,1] \mid f(0) = c\}. \end{aligned}$$

Claim: $V(c)$ is a vector subspace of $C[0,1]$ if and only if $c = 0$.
Proof: $(\Rightarrow)$. Assume $V(c)$ is a subspace of $C[0,1]$. Then $V(c)$ is closed under addition. Let $h(x) = f(x) + g(x)$. We have $h(0) = f(0) + g(0) = c + c = 2c$, but $h(x) \in V(c)$, so $h(0) = 0$. This implies $2c = c$ which implies $c = 0$.

$(\Leftarrow)$. We need to check that the vector space is closed with respect to addition and scalar multiplication. Let $f, g \in V(0)$. Let $h = f + g$. $h(0) = f(0) + g(0) = 0 + 0 = 0$, so $V(0)$ is closed with respect to addition. Now, Let $h = \lambda f$. Then, $h(0) = \lambda f(0) = \lambda \cdot 0 = 0$, so $h(0)$ is closed with respect to scalar multiplication. (Note that we're inheriting that $h \in C[0,1]$ because $C[0,1]$ is a vector space.)

I took off 2 points if you did not show that $V(0)$ is a subspace by checking both conditions.

6

There are many examples of infinite sets of linearly independent vectors in $C[0,1]$:

$$\begin{aligned}
S_a &= \{\sin(\pi 2^n x) \mid n \in \{0,1,2,\dots\}\} \\
S_b &= \{\cos(\pi n x) \mid n \in \mathbb{N}\} \\
S_c &= \{e^{nx} \mid n \in \mathbb{N}\} \\
S_d &= \{\log(n(x+1)) \mid n \in \mathbb{N}\} \\
S_e &= \left\{ f_n(x) \mid n \in \mathbb{N}, f_n(x) = \begin{cases} 0 & 0 \le x < \frac{1}{n+1} \\ x - \frac{1}{n+1} & \frac{1}{n+1} \le x < \frac{1}{2}\left(\frac{1}{n+1} + \frac{1}{n}\right) \\ -x + \frac{1}{n} & \frac{1}{2}\left(\frac{1}{n+1} + \frac{1}{n}\right) \le x < \frac{1}{n} \\ 0 & x \ge \frac{1}{n}. \end{cases} \right\} \\
S_f &= \left\{ f_n(x) \mid n \in \mathbb{N}, n \ge 3, f_n(x) = \begin{cases} 1 & x < \frac{1}{n} \\ 1 - n(x - 1/n) & \frac{1}{n} \le x \le \frac{2}{n} \\ 0 & x > \frac{2}{n} \end{cases} \right\}
\end{aligned}$$

Giving any of the above sets, or many similar ones, gave you seven out of ten points. The last three points could be earned by proofs of varying detail.

Proofs:
All the above sets are infinite. They are all continuous either because they are compositions of continuous functions or, for the piecewise functions, because the limits of the joints work out to the same values. It really helps to draw some of the functions to see what's going on; it also helps the grader if you're using ugly piecewise functions.

To show that an infinite set of vectors is linearly independent, you must show that any finite subset of them is linearly independent. Since in our examples all the sets are countable, it is sufficient (do you know why?) to show that the first n vectors are linearly independent.

Important note: It is not enough to show that vectors are pairwise independent! Consider the set $S = \{(1,0),(1,1),(0,1)\} \subset \mathbb{R}^2$. Any pair in S is linearly independent, but clearly $(1,1) = (1,0)+(0,1)$.

a. Note that $\sin(\pi 2^n x)$, restricted to $x \in [0,1]$ has a root at $\frac{1}{2^n}$.

$$c_1 \sin(\pi x) + c_2 \sin(\pi 2x) + c_3 \sin(\pi 4x) + \cdots + c_n \sin(\pi 2^n x) = 0.$$

Then, at $x = \frac{1}{2}$, all $\sin(\pi 2^n \frac{1}{2}) = 0$ when $n > 0$, so we must have $c_1 \sin(\pi \frac{1}{2}) = 0$, which implies that $c_1 = 0$. Then, at $x = \frac{1}{4}$, we find that all but the first two terms on the left-hand side are zero, and the first term is 0 because $c_1 = 0$, so we conclude that $c_2 = 0$. We can continue this process another $n-2$ times to find that $c_i = 0$ for all $i \in \{0,1,\ldots,n\}$. This shows that S_1 is linearly independent.

b. A similar proof can be cooked up for cos. You can also use an argument about the period of the sum of n vectors, but that's harder to solidify.

c. To prove that S_3 is linearly independent, we can use the subsititution $y = e^x$, in which case we are dealing with a set

$$S' = \{y^n \mid n \in \mathbb{N}\},$$

where all the functions are to be regarded as $x^n : [1,e] \to \mathbb{R}$. Then, suppose

$$c_1 x^1 + c_2 x^2 + \cdots + c_n x^n = 0.$$

Suppose, WLOG, $c_n \neq 0$. The function on the left hand side has at most n roots on the real line, which implies that it has at most n roots on $[1,e]$. The zero function has an uncountable number of roots on $[1,e]$, so they cannot be the same function.

If you did not mention that you were doing a substitution, you did not receive credit for this proof. You have to justify why e^{nx} is "like" x^n.

d. Suppose

$$c_1 \log(x+1) + c_2 \log(2(x+1)) + \cdots + c_n \log(n(x+1)) = 0.$$

Using the properties of logarithms and applying the exponential function to both sides we get:

$$(x+1)^{c_1} 2^{c_2} (x+1)^{c_2} \ldots n^{c_n} (x+1)^{c_n} = 1.$$

This implies that $c_i = 1$ for $i \in \mathbb{N}$. (People trying to show this by induction usually failed because they misused the induction hypothesis.)

e. Every function in this set is zero except between $1/(n+1)$ and $1/n$. The important thing to note that here is that $f_i(\frac{1}{2}\left(\frac{1}{n} + \frac{1}{n+1}\right)) \neq 0$ if and only if $n = i$. So, like in the first example with $\sin(\pi n x)$, we write a linear combination, choose $x = (.5)(1/1 + 1/2)$ and find $c_1 = 0$. Then take the next $(.5)(1/n + 1/(n+1))$ and find $c_2 = 0$, and so on.

f. Similar to previous one. Choose $x = 2/2, 2/3, 2/4, \ldots$ and show that $c_1 = c_2 = \cdots = c_n = 0$.