

## MATH 23A SOLUTION SET #3 (PART C)

ISIDORA MILIN

**Problem (7).** Show that if  $V$  is a vector space with  $\dim(V) = n$ , then any collection of  $n + 1$  vectors in  $V$  is linearly dependent.

*Proof 1.* One way to solve this problem is to assume the following theorem done in class:

**Theorem.** If  $V$  is an  $n$ -dimensional vector space, then any set of  $n$  linearly independent vectors in  $V$  is a basis for  $V$ .

Note that even though the version of the proof of this theorem which was done in class used the claim of this problem, you were allowed to use the theorem in your solution.

So let  $v_1, v_2, \dots, v_{n+1}$  be any  $n + 1$  vectors in  $V$ . Suppose first that  $v_1, \dots, v_n$  are linearly dependent. Then, there exist scalars  $a_1, \dots, a_n \in F$ , not all 0, such that:

$$\begin{aligned} a_1 v_1 + \dots + a_n v_n &= \vec{0} \\ \Rightarrow a_1 v_1 + \dots + a_n v_n + 0 \cdot v_{n+1} &= \vec{0} \end{aligned}$$

where not all  $a_1, \dots, a_n$  zero, and hence  $v_1, \dots, v_n, v_{n+1}$  must be linearly dependent as well.

Now suppose that  $v_1, \dots, v_n$  are linearly independent. Then, by the theorem above they form a basis for  $V$ . In particular, they span  $V$ . But then  $v_{n+1} \in V$  is in their span as well, so that  $\exists b_1, \dots, b_n \in F$  such that:

$$\begin{aligned} v_{n+1} &= b_1 v_1 + \dots + b_n v_n \\ \Rightarrow b_1 v_1 + \dots + b_n v_n + (-1)v_{n+1} &= \vec{0} \end{aligned}$$

so that  $v_1, \dots, v_n, v_{n+1}$  are linearly dependent. □

*Proof 2.* Another way to solve this problem is by induction on  $n$ . As many of you have noticed, this is how it is done in your linear algebra textbook (see pages 34-35 of Curtis, where a slightly more general claim is proven).

**Base Case.** When  $n = 1$ , let  $\{v\}$  be a basis for  $V$ , and  $v_1, v_2 \in V$  be any two vectors in  $V$ . Then,  $\exists a_1, a_2 \in F$  such that  $v_1 = a_1 v$  and  $v_2 = a_2 v$ . If  $a_1 a_2 = 0$ , then either  $v_1 = \vec{0}$  or  $v_2 = \vec{0}$ , which, in both cases, makes  $\{v_1, v_2\}$  into a linearly dependent set. Otherwise, we can write  $\frac{1}{a_1} v_1 - \frac{1}{a_2} v_2 = \vec{0}$ , so that  $\{v_1, v_2\}$  is again linearly dependent.

**Inductive Step.** Suppose that in any  $n - 1$ -dimensional vector space any  $n$  vectors are linearly dependent. We want to show this implies that any

$n + 1$  vectors  $v_1, v_2, \dots, v_n$  in any  $n$ -dimensional vector space  $V$  are linearly dependant.

Let  $e_1, \dots, e_n$  be a basis for  $V$ . Then, all vectors in  $V$  are expressible as linear combinations of the  $e_i$ 's, so we can find scalars  $a_{i,j}$ , where  $1 \leq i \leq n+1$ ,  $1 \leq j \leq n$  such that:

$$\begin{aligned} v_1 &= a_{1,1}e_1 + a_{1,2}e_2 + \dots + a_{1,n}e_n \\ v_2 &= a_{2,1}e_1 + a_{2,2}e_2 + \dots + a_{2,n}e_n \\ &\dots \\ v_n &= a_{n,1}e_1 + a_{n,2}e_2 + \dots + a_{n,n}e_n \\ v_{n+1} &= a_{n+1,1}e_1 + a_{n+1,2}e_2 + \dots + a_{n+1,n}e_n \end{aligned}$$

Now, consider the scalars "in the first column" i.e.  $a_{1,1}, a_{2,1}, \dots, a_{n,1}, a_{n+1,1}$ . If all of them are 0, then we have that  $v_1, v_2, \dots, v_n, v_{n+1} \in V' = \text{span}(e_2, \dots, e_n)$ . Now,  $\dim(V') = n - 1$  and  $v_1, \dots, v_n \in V'$ , so by the inductive hypothesis,  $v_1 \dots v_n$  are linearly dependent, which implies  $v_1, \dots, v_n, v_{n+1}$  are linearly dependent as well.

So we now assume that  $\exists a_{i,1} \neq 0$ . WLOG (i.e. by reordering the  $v_i$ 's), assume  $a_{1,1} \neq 0$ . Now consider the following  $n$  vectors:

$$\begin{aligned} v'_2 &= v_2 - \frac{a_{2,1}}{a_{1,1}}v_1 = a'_{2,2}e_2 + \dots + a'_{2,n}e_n \\ &\dots \\ v'_n &= v_n - \frac{a_{n,1}}{a_{1,1}}v_1 = a'_{n,2}e_2 + \dots + a'_{n,n}e_n \\ v'_{n+1} &= v_{n+1} - \frac{a_{n+1,1}}{a_{1,1}}v_1 = a'_{n+1,2}e_2 + \dots + a'_{n+1,n}e_n \end{aligned}$$

Note that  $v'_2, \dots, v'_n, v'_{n+1}$  are  $n$  vectors in the  $n - 1$ -dimensional vector space  $V' = \text{span}(e_2, \dots, e_n)$ , so that by the inductive hypothesis, they are linearly dependent. That is,  $\exists \lambda_2, \dots, \lambda_{n+1} \in F$ , not all zero, such that

$$\lambda_2 v'_2 + \dots + \lambda_n v'_n + \lambda_{n+1} v'_{n+1} = \vec{0}$$

This implies that:

$$\lambda_2 \left( v_2 - \frac{a_{2,1}}{a_{1,1}}v_1 \right) + \dots + \lambda_n \left( v_n - \frac{a_{n,1}}{a_{1,1}}v_1 \right) + \lambda_{n+1} \left( v_{n+1} - \frac{a_{n+1,1}}{a_{1,1}}v_1 \right) = \vec{0}$$

and we finally get:

$$-\left( \lambda_2 \frac{a_{2,1}}{a_{1,1}} + \dots + \lambda_n \frac{a_{n,1}}{a_{1,1}} + \lambda_{n+1} \frac{a_{n+1,1}}{a_{1,1}} \right) v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n + \lambda_{n+1} v_{n+1} = \vec{0}$$

Thus,  $v_1, v_2, \dots, v_n, v_{n+1}$  are linearly dependent, which ends the proof of the inductive step.  $\square$

Note that the above proof is not at all trivial, and hence just citing it on the homework was not sufficient to get full credit for this problem. In particular, it was not sufficient to persuade me that you have read it and fully understood it, let alone that you would, after reading it, be capable of writing it up *in your own words*.

A few people used in their proofs the claim that a system of  $n$  linear equations with  $n + 1$  unknowns has infinitely many solutions. This was not acceptable, since such theorems about systems of linear equations are based on the corresponding theorems about vector spaces. Namely, the space of all solutions to a system of linear equation turns out to be a subspace of the entire domain space considered as a vector space, and you are likely to learn more about how this works later in the course.

To my chagrin, surprisingly few people heeded an extremely helpful hint for this problem that was given (admittedly somewhat late) in lecture. The following proof, which is based on that hint, is sooo nice that it would also work to fill the gap in the proof that the dimension of a vector space is well-defined (i.e. that any two bases of a finite-dimensional vector space have the same cardinality), so please make sure you understand it! And ask me or one of the other CA's to explain it to you if you don't! If you are still thirsty for knowledge, *Linear Algebra Done Right* by S. Axler has a nice, brief exposition of linear independence, spanning sets and related concepts (see p.22-30), and by reading it, you can also review for the midterm other linear algebra topics we covered so far (see p.2-62).

*Proof 3.* Here we show that any list of linearly independent vectors in  $V$  is shorter than any list of vectors that span  $V$ . This is sufficient to solve the problem, since if  $\dim(V) = n$ ,  $V$  has a basis consisting of  $n$  vectors in  $V$ , and those  $n$  vectors span  $V$ , so that any list of linearly independent vectors in  $V$  cannot contain more than  $n$  vectors.

So suppose that  $A = \{u_1, \dots, u_{n+1}\}$  is linearly independent in  $V$  and that  $B = \{v_1, \dots, v_n\}$  spans  $V$ . Now consider the following procedure, which, at each step, modifies  $B$  by throwing out one of the  $v_i$ 's and adding one of the  $u_j$ 's:

### Step 1

The list  $B = \{v_1, \dots, v_n\}$  spans  $V$ , so that  $u_1 \in V$  is a linear combination of the elements of  $B$ . Thus, the set  $B \cup u_1 = \{u_1, v_1, \dots, v_n\}$  is linearly dependent. Namely, there exist scalars  $a_1, \dots, a_n \in F$  such that  $u_1 = a_1 v_1 + \dots + a_n v_n$ . Not all  $a_1, \dots, a_n$  can be 0, for then we would have  $u_1 = \vec{0}$ , and the set  $\{u_1, \dots, u_n\}$  would be linearly dependent, since for  $\lambda \neq 0$  we would have that  $\lambda u_1 + 0 \cdot u_2 + \dots + 0 \cdot u_m = \vec{0}$ . Thus,  $-u_1 + a_1 v_1 + \dots + a_n v_n = \vec{0}$  holds, where at least one  $a_i \neq 0$ . Let  $i \in \{1, \dots, n\}$  be the largest index for which  $a_i \neq 0$ . Then we get that

$$v_i = \frac{1}{a_i} u_1 - \frac{a_1}{a_i} v_1 - \frac{a_2}{a_i} v_2 - \dots - \frac{a_{i-1}}{a_i} v_{i-1}$$

and we claim that  $B_1 = \{u_1, v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n\}$  is a new list of  $n$  vectors in  $V$  which again spans  $V$ . To see that any vector in  $V$  is a linear combination of vectors in  $B_1$ , use the fact that  $B$  spanned  $V$ , so that any vector in  $V$  is a linear combination of  $v_i$ 's and substitute in the expression for  $v_i$  given above:

$$\begin{aligned} v &= \lambda_1 v_1 + \dots + \lambda_n v_n \\ &= \lambda_1 v_1 + \dots + \lambda_{i-1} v_{i-1} + \lambda_i \left( \frac{1}{a_i} u_1 - \frac{a_1}{a_i} v_1 - \frac{a_2}{a_i} v_2 - \dots - \frac{a_{i-1}}{a_i} v_{i-1} \right) \\ &\quad + \lambda_{i+1} v_{i+1} + \dots + \lambda_n v_n \end{aligned}$$

Thus, we have obtained a new spanning set  $B_1 = \{u_1, w_1, \dots, w_{n-1}\}$  of  $V$ , again of length  $n$ , where  $w_j$ 's are just relabeled  $v_j$ 's that haven't been thrown out (i.e. they just stand for  $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n$ ).

### Step i

Suppose that in the step  $i - 1$  we have obtained a list of  $n$  vectors in  $V$ ,  $B_{i-1} = \{u_1, \dots, u_{i-1}, w_1, \dots, w_{n-i+1}\}$  which spans  $V$ . Then,  $u_i \in V$  is a linear combination of the elements of  $B_{i-1}$ , so that there exist scalars  $a_1, \dots, a_n \in F$ , not all 0 (since  $u_i \neq \vec{0}$ ), such that

$$a_1 u_1 + \dots + a_{i-1} u_{i-1} - u_i + a_i w_1 + a_{i+1} w_2 + \dots + a_n w_{n-i+1} = \vec{0}$$

Let  $j \in \{1, \dots, n\}$  be the largest index for which  $a_j \neq 0$ . Notice that  $j \geq i$  since otherwise we would have  $a_i = a_{i+1} = \dots = a_n = 0$  and hence  $a_1 u_1 + \dots + a_{i-1} u_{i-1} = u_i$ , which would contradict the linear independence of  $A = \{u_1, \dots, u_{n+1}\}$ . For this  $j$  we have that:

$$w_j = -\frac{a_1}{a_j} u_1 - \dots - \frac{a_{i-1}}{a_j} u_{i-1} + \frac{1}{a_j} u_i - \frac{a_i}{a_j} w_1 - \frac{a_{i+1}}{a_j} w_2 - \dots - \frac{a_{j-1}}{a_j} w_{j-1}$$

Now, throw out  $w_j$  from  $B_{i-1}$  and throw in  $u_i$  instead, to get:

$$B_i = \{u_1, \dots, u_{i-1}, u_i, w_1, \dots, w_{j-1}, w_{j+1}, \dots, w_{n-i+1}\}$$

This new list of  $n$  vectors in  $V$  again spans  $V$ . This is because  $B_{i-1}$  spanned  $V$  and any linear combination of elements of  $B_{i-1}$  becomes a linear combination of elements of  $B_i$  after substituting in it the above expression for  $w_j$ .

**Conclusion** After running this procedure through the first  $n$  steps, we conclude that  $B_n = \{u_1, \dots, u_n\} \subseteq A$  spans  $V$ . Hence,  $u_{n+1}$  is a linear combination of  $u_1, \dots, u_n$ , which contradicts our assumption that  $u_1, \dots, u_{n+1}$  were linearly independent. It follows that our assumption must have been false!  $\square$

### Notes

When writing about linear independence, remember that saying something like "Vector  $v$  is linearly independent" implies simply that  $v \neq \vec{0}$ . To be even pickier about it, you should really say: "The set  $\{v\}$  is linearly independent." However, neither of the two sentences expresses that  $v$  is linearly independent "from" some vectors  $w_1, \dots, w_n$ , that is, that  $v$  is not a

linear combination of  $w_1, \dots, w_n$ . Many people assumed that if I read a sentence "Vector  $v$  is linearly independent" I should be able to figure out that they really meant "not a linear combination of these  $w_1, \dots, w_n$  that have been mentioned 100 times previously", and that if I read a sentence "vector  $v$  is linearly dependent", I would figure out that they meant that  $v \in \text{Span}(w_1, \dots, w_n)$ . Well, I couldn't figure it out. So, when writing up your solutions, keep in mind that Isidora (and probably the other CA's as well) does not have a sixth sense to read your mind with it!

**Problem (8).** Let  $S$  be some set with finite cardinality  $n$ . For simplicity, assume  $S = \{1, 2, \dots, n\}$ . Find a basis for the vector space of functions  $V = \{f : S \rightarrow \mathbb{R}\}$ , where addition and scalar multiplication are defined as for functions of one real variable.

*Solution.* For all  $i \in \{1, 2, \dots, n\}$ , define  $f_i \in V$  as follows:

$$f_i(j) = \begin{cases} 1, & \text{for } i = j \\ 0, & \text{for } i \neq j \end{cases}$$

We claim that  $\{f_1, \dots, f_n\}$  is a basis for  $V$ . To show this, we need to check that  $f_1, \dots, f_n$  are linearly independent, and that they span  $V$ .

**Linear Independence.** Let  $a_1, \dots, a_n \in \mathbb{R}$  be such that  $a_1 f_1 + \dots + a_n f_n = f_0$ , where  $f_0$  here denotes the zero-function, taking value 0 on its entire domain  $S$ . We want to show that this implies  $a_1 = \dots = a_n = 0$ . But  $f_0(j) = 0$  for all  $j \in S$ , and moreover,  $f_i(j) = 0$  for all  $f_i$ 's except  $f_j$ , so that:

$$0 = f_0(j) = a_1 f_1(j) + \dots + a_n f_n(j) = a_j f_j(j) = a_j$$

This works for all  $j$ , so we have that  $a_1 = \dots = a_n = 0$  and  $f_1, \dots, f_n$  are linearly independent.

**Span.** Let  $f \in V$  be any function from  $S$  to  $\mathbb{R}$ , and let  $f(j) = a_j \in \mathbb{R}$ , for all  $j \in S$ . As we have seen above, we have that:

$$(a_1 f_1 + \dots + a_n f_n)(j) = a_1 f_1(j) + \dots + a_n f_n(j) = a_j f_j(j) = a_j$$

Thus, the functions  $f$  and  $a_1 f_1 + \dots + a_n f_n$  agree on all points of  $S$ , so that we conclude that  $f = f(1)f_1 + \dots + f(n)f_n$  and hence  $f_1, \dots, f_n$  spans  $V$ .  $\square$

If this proof doesn't make you happy because you are still wondering how would one come up with these  $f_i$ 's out of the blue, notice that  $V$  can be identified with  $\mathbb{R}^n$  via the following vector space isomorphism:

$$T : V \rightarrow \mathbb{R}^n, f \mapsto (f(1), f(2), \dots, f(n))$$

It is not difficult to check that  $T$  is a bijective linear map (Do it!). So to find a basis for  $V$ , we can take the standard basis for  $\mathbb{R}^n$ , namely:

$$B = \{e_1 = (1, 0, \dots, 0), e_2 = (0, 1, \dots, 0), \dots, e_n = (0, 0, \dots, 1)\}$$

and see which vectors in  $V$  get mapped under  $T$  to vectors in  $B$ . It turns out that  $T(f_i) = e_i$  for precisely those  $f_i$ 's we considered above.