MATH 23b, SPRING 2003
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Homework Assignment # 4
Due: February 28, 2003

Homework Assignment #4 (Final Version)

1. Read Fitzpartick, Sections 14.2, 15.2 amd 15.3, and the statements of the Inverse Function Theorem (16.1 and 16.3) and Dini's Theorem (17.1) and the Generalized Implicit Function Theorem (17.2). In particular, we will be discussing the Inverse Function Theorem and Implicit Function Theorem (though not quite as Fitpatrick does) before using them to help classify extrema of functions.

2. (A) (previously known as #3.3)

   Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function with continuous second-order partial derivatives (so that, in particular, Theorem 13.3 applies, and $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$).

   With $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ the usual gradient of f, we make the following definitions:

   - $||\nabla f||^2 = (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2$ is the norm (squared) of the gradient of f.
   - $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$ is the *Laplacian* of f.

   Finally, let $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by $g(r, \theta) = f(r \cos\theta, r \sin\theta)$.

   (a) Show that $||\nabla f||^2 = (\frac{\partial g}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial g}{\partial \theta})^2$.

   (b) Show that $\nabla^2 f = \frac{\partial^2 g}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2} + \frac{1}{r}\frac{\partial g}{\partial r}$.

3. (B) (borrowed from Fitzpatrick, ex. 15.21 and probs. 15.3.6 and 15.3.9)

   Following up on problem #4.2, if $f : \mathbb{R}^n \rightarrow \mathbb{R}$ has continuous second-order partials, the *Laplacian* of f is defined to be $\nabla^2 f = \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2}$.

   With f as above, we say that f is *harmonic* on the open set $U \subset \mathbb{R}^n$ provided that $\nabla^2 f(\mathbf{x}) = 0, \ \ \forall \mathbf{x} \in U$.

   (a) Find a (simple) condition on the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $f(x, y) = ax^2 + bxy + cy^2 + dx + ey + k$ that makes f harmonic.

   (b) Show that $f : \mathbb{R}^n \rightarrow \mathbb{R}$ defined by $f(\mathbf{x}) = \frac{1}{||\mathbf{x}||^{n-2}}$ is harmonic on $U = \mathbb{R}^n \setminus \{\mathbf{0}\}$.

   (c) Show that if $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ is harmonic, then $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $f(x, y) = g(e^x \cos y, e^x \sin y)$ is also harmonic.

4. (deferred)

   Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be given by $f(x,y) = (\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$. Show that f is locally invertible in a neighborhood of every point except the origin, and compute f^{-1} explicitly.

5. (C) Consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by:

$$f(x) = \begin{cases} \frac{x}{2} + x^2 \sin(\frac{1}{x}) & , \text{ if } x \neq 0 \\ 0 & , \text{ if } x = 0 \end{cases}$$

   (a) Show that f is differentiable at 0 and that $f'(0) = \frac{1}{2}$.

   (b) Show that there is no open set (interval) containing 0 on which f is one-to-one.

6. (D) In the following problem, we consider the notion of the *local invertibility* of a function and the relationship between this condition and that of injectivity.

   (a) Read the introduction to Section 16.1.

   (b) Suppose that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is locally invertible at every point of its domain (that is, $f'(a) \neq 0, \forall a \in \mathbb{R}$). Show that f is one-to-one.

   (c) Consider $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by $g(x,y) = (e^x \cos y, e^x \sin y)$. Show that g is locally invertible at every point of its domain (that is, $\det[Jg(\mathbf{x})] \neq 0, \forall \mathbf{x} \in \mathbb{R}^2$), but that g is not one-to-one.

7. (Not required)

   In class, we mentioned the existence of functions that were differentiable at a point but not on any open set containing that point. As an example, consider the function $f_3 : \mathbb{R} \to \mathbb{R}$ given by:

$$f_3(x) = \begin{cases} 0 & , \text{ if } x \notin \mathbb{Q} \\ \frac{1}{q^3} & , \text{ if } x = \frac{p}{q} \text{ in lowest terms} \end{cases}$$

   (a) Show that f_3 is not continuous at any $x \in \mathbb{Q}$.

   (b) Show that f_3 is differentiable at $x = \sqrt{2}$.