Math 23b, Spring 2003

Problem Set 4, Part A
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Problem 3: *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function with continuous second-order partial derivatives. Show that:*

(a) $||\nabla f||^2 = \left(\frac{\partial g}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial g}{\partial \theta}\right)^2$

(b) $\nabla^2 f = \frac{\partial^2 g}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2} + \frac{1}{r}\frac{\partial g}{\partial r}$

Proof. (a) We have $g(r,\theta) = f(r\cos\theta, r\sin\theta) = f(x,y)$. Thus applying the chain rule we get

$$\frac{\partial g}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta.$$

Similarly,

$$\frac{\partial g}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = -\frac{\partial f}{\partial x}r\sin\theta + \frac{\partial f}{\partial y}r\cos\theta.$$

Using these expressions for $\frac{\partial g}{\partial r}$ and $\frac{\partial g}{\partial \theta}$ we directly calculate that

$$\left(\frac{\partial g}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial g}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = ||\nabla f||^2.$$

(b) Applying again the chain rule we obtain:

$$\frac{\partial}{\partial r}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial^2 f}{\partial y \partial x}\frac{\partial y}{\partial r}.$$

Similarly,

$$\frac{\partial}{\partial \theta}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial \theta} + \frac{\partial^2 f}{\partial y \partial x}\frac{\partial y}{\partial \theta},$$

$$\frac{\partial}{\partial r}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial r} + \frac{\partial^2 f}{\partial x \partial y}\frac{\partial x}{\partial r},$$

$$\frac{\partial}{\partial \theta}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial \theta} + \frac{\partial^2 f}{\partial x \partial y}\frac{\partial x}{\partial \theta}.$$

Combing these with the product rule we get:

$$\frac{\partial^2 g}{\partial r^2} = \frac{\partial^2 f}{\partial x^2}\cos^2\theta + \frac{\partial^2 f}{\partial y \partial x}\cos\theta\sin\theta + \frac{\partial^2 f}{\partial y^2}\sin^2\theta + \frac{\partial^2 f}{\partial x \partial y}\sin\theta\cos\theta,$$

$$\frac{\partial^2 g}{\partial \theta^2} = \left(\frac{\partial^2 f}{\partial x^2} r^2 \sin^2\theta - \frac{\partial^2 f}{\partial y \partial x} r^2 \cos\theta\sin\theta - \frac{\partial f}{\partial x} r\cos\theta\right) +$$

$$+ \left(\frac{\partial^2 f}{\partial y^2} r^2 \cos^2\theta - \frac{\partial^2 f}{\partial x \partial y} r^2 \sin\theta\cos\theta - \frac{\partial f}{\partial y} r\sin\theta\right).$$

Plugging in what we got for $\frac{\partial^2 g}{\partial r^2}$, $\frac{\partial^2 g}{\partial \theta^2}$, and $\frac{\partial g}{\partial r}$ and enjoying the cancellation we get that $\frac{\partial^2 g}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2} + \frac{1}{r}\frac{\partial g}{\partial r} = \nabla^2 f$ and we are done with this problem.

□

Important Remark: The techniques involved in this solution are basic but extremely important. That is why everyone who did not get a perfect score on this problem should read the solution and understand it completely.