Solution Set 4C

Math 23b
March 6, 2003

5. a) By definition, the derivative of f at 0 is:

$$\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim_{h\to 0}\frac{\frac{h}{2}+h^2\sin\frac{1}{h}}{h}=\lim_{h\to 0}\left(\frac{1}{2}+h\sin\frac{1}{h}\right)=\frac{1}{2}$$

because $\sin(1/h)$ is bounded, so $h\sin(1/h)\to 0$ as $h\to 0$ by the squeeze theorem.

b) Let $U\subset\mathbf{R}$ be an open set containing 0, and choose ϵ small enough that $(0,\epsilon)\subset U$. We will show that f is not one-to-one on $(0,\epsilon)$. First note that when $x\neq 0$, the derivative is given by

$$f'(x)=\frac{1}{2}+2x\sin\frac{1}{x}-\cos\frac{1}{x}$$

which is differentiable. Choose an $n\in\mathbf{N}$ such that $1/2\pi n<\epsilon$. We have

$$f'\left(\frac{1}{2\pi n}\right)=-\frac{1}{2}\qquad f'\left(\frac{1}{2\pi n+\pi}\right)=\frac{3}{2}.$$

Since f' is continuous, there is some c between $1/(2\pi n+\pi)$ and $1/2\pi n$ such that $f'(c)=0$. Since the sign of f' changes, we can assume that $f''(c)\neq 0$, so that c is a local extremum of f. Therefore f is not one-to-one in any neighborhood of c, so we are done.

Notes on this problem:

(1) Many people made the same argument for part (b) above, but they only claimed that one has to find an x such that $f'(x)<0$ because $f'(0)>0$, and f must be monotonically increasing or decreasing. However, f' is not continuous at 0, so this argument is quite difficult to make — one can not find a local minimum or maximum in the same way as when f' is continuous.

In fact, I believe that if f is not $\mathcal{C}^1$, then in general it is false that if the derivative changes signs then f has a local maximum or minimum. For instance, it is possible to construct convoluted functions that have zero derivative almost everywhere but are still monotonically increasing. Or, I would argue as follows: you could have a function f whose derivative was negative for $x\neq 0$ but which had a jump discontinuity and had positive derivative only at $x=0$. Then if you use the fundamental theorem of calculus to get the function back from its derivative (assuming you can do this), you'll have a monotonically decreasing function — the one point at which it has positive derivative doesn't contribute to the integral.

The point is, I'm not convinced that you can simply say that it suffices to find an x with $f'(x)<0$.

(2) If you have a function $f:\mathbf{R}\to\mathbf{R}$, then the standard single-variable calculus definition

of the derivative is *equivalent* to the one given in class, so use the former! It's much easier.

(3) Note that this problem is a counter-example to the inverse function theorem when f is not $\mathcal{C}^1$!

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