

MATH 23a, FALL 2002  
THEORETICAL LINEAR ALGEBRA  
AND MULTIVARIABLE CALCULUS  
(Final Version) Homework Assignment #5  
Due: October 25, 2002

1. Read Sections 11–13 (Chapter 3) and 22–23 (Chapter 7) of Curtis.
2. (A) In this problem, we consider the *shift* operator. Consider the linear map  $S : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  which acts as follows:

$$S(x, y, z) = (0, x, y).$$

Find the kernel and image of  $S$ , and verify that

$$\dim(\text{Ker}(S)) + \dim(\text{Im}(S)) = \dim(\mathbb{R}^3).$$

3. (A) We generalize the notion of the *shift* operator. Let  $V$  be the vector space of all infinite sequences of real numbers as in Problem #3.9, and consider the linear maps  $S : V \rightarrow V$  and  $T : V \rightarrow V$ , where  $S$  and  $T$  act as follows:

$$S(a_0, a_1, a_2, \dots) = (0, a_0, a_1, a_2, \dots)$$

$$T(a_0, a_1, a_2, \dots) = (a_1, a_2, \dots)$$

- (a) Find the kernel and image of  $S$ . How does the result about the dimensions of kernels and images apply?
  - (b) Show that  $T \circ S = I$  but that  $S \circ T \neq I$ , where  $I : V \rightarrow V$  is the identity map.
  - (c) Which of  $S$  and  $T$  is onto? Which is one-to-one? Which is invertible? Explain.
4. (B) Consider the linear differential operator  $D : C^\infty \rightarrow C^\infty$  given by  $D(f) = f' + af$ , where  $a$  is some fixed real number. (Recall that  $C^\infty$  is the vector space of all functions which are infinitely-differentiable.)
  - (a) Find  $\text{Ker}(D)$ .
  - (b) Show that  $D$  is surjective. (Hint: Let  $g \in C^\infty$ , and show that  $f' + af = g$  has a solution. Multiply both sides by the *integrating factor*  $e^{ax}$ , and integrate both sides—on the left using the product rule!)

5. (C) Let  $A : U \rightarrow V$  and  $B : V \rightarrow W$  be linear maps between finite-dimensional vector spaces.
- (a) Show that if  $A$  and  $B$  are both surjective, then so is  $B \circ A$ .
  - (b) Show that if  $B$  is one-to-one, then  $\text{Ker}(B \circ A) = \text{Ker}(A)$ .
  - (c) Assuming  $A$  and  $B$  are surjective, show that  $\dim(\text{Ker}(B \circ A)) = \dim(\text{Ker}(A)) + \dim(\text{Ker}(B))$ .
6. (D) Find the inverse of the linear map  $L : (\mathbb{Z}/7\mathbb{Z})^3 \rightarrow (\mathbb{Z}/7\mathbb{Z})^3$  given by  $L(x, y, z) = (x + y + z, 2x + 3y + 4z, 3x + 4y + 6z)$ .
7. (D) Let  $F$  be any field. Show that the linear operator  $L : F^2 \rightarrow F^2$  given by  $L(x, y) = (ax + by, cx + dy)$  is invertible if and only if  $ad - bc \neq 0$ .
8. (Not required) Show that if  $A : U \rightarrow V$  and  $B : V \rightarrow W$  are both invertible linear maps, then  $(BA)^{-1} = A^{-1}B^{-1}$ .