Solutions for Homework 5, Part C

Well, most of you actually did pretty well on this problem; the median score was a 9.5 out of 10 and even the mean was a 9.1!

4. By the Implicit Function Theorem, it follows that if f is a continuously differentiable function from an open set U in $\mathbb{R}^3$ to $\mathbb{R}$, (x, y, z) is a point in U such that $f(x, y, z) = 0$, and $\frac{\partial f}{\partial z} \neq 0$ (as the determinant of a 1×1 matrix is the entry of that matrix), then f defines z implicitly as a function of x and y near the point (x, y, z).

Therefore, because $\sin(x+z)+\log(yz^2)$ is clearly continuously differentiable in a neighborhood of $(1, 1, 1)$ (in particular, the neighborhood consisting of all points (x, y, z) in $\mathbb{R}^3$ where $y > 0$ and $z < 0$), $\sin(x + z) + \log(yz^2) = \sin(1+(-1))+\log(1*(-1)^2) = 0+0 = 0$ when $(x, y, z) = (1, 1, -1)$, $e^{(x+z)} + yz$ is continuously differentiable in all of $\mathbb{R}^3$ and $e^{(x+z)} + yz = e^{1+(-1)} + 1*(-1) = 1 + (-1) = 0$ when $(x, y, z) = (1, 1, -1)$, it suffices to show that $\frac{\partial f}{\partial z} \neq 0$ at the point $(1, 1, -1)$ when $f(x, y, z) = \sin(x+z)+\log(yz^2)$ to show that $\sin(x+z)+\log(yz^2) = 0$ defines z implicitly as a function of x and y near the point $(1, 1, -1)$. Similarly, it suffices to show that $\frac{\partial g}{\partial z} \neq 0$ when $g(x, y, z) = e^{(x+z)} + yz$ at the point $(1, 1, -1)$ to show that $e^{(x+z)} + yz = 0$ defines z implicitly as a function of x and y near the point $(1, 1, -1)$.

However, $\frac{\partial f}{\partial z} = \cos(x + z) + 2yz/(yz^2) = \cos(x + z) + 2/z$, which equals $\cos(1 + (-1)) + 2/(-1) = -1 \neq 0$ at the point $(1, 1, -1)$.

Similarly, $\frac{\partial g}{\partial z} = e^{(x+z)} + y$, which equals $e^{1+(-1)} + 1 = 2 \neq 0$ at the point $(1, 1, -1)$; therefore, both of the equations listed in the problem statement define z implicitly as a function of x and y near the point $(1, 1, -1)$.

NOTE: The most popular mistake on this problem was a misinterpretation of what "log" means. A large number of people went by the calculator's definition and interpreted "*log*" to be a base-10 logarithm. However, this is not standard mathematical notation. Mathematicians use "log" to refer to the natural logarithm (in base e), what calculators call "ln". Making this mistake cost half a point; if you want to use a base-10 logarithm you must say "$\log_{10}$".

A few people had trouble differentiating the logarithmic expression; the Chain Rule states that if $f(x) = \log(g(x))$ where $g(x)$ is a differentiable function, then $f'(x) = 1/(g(x)) * g'(x) = \frac{g'(x)}{g(x)}$. Differentiation mistakes usually cost a single point out of 10.

And finally, because this problem was an application of the Implicit Function Theorem, failure to state the name of the theorem cost you two points and referring to the theorem by the wrong name (e.g. the "Implicit Value Theorem") cost you one point.