Solution Set 5D

Math 23b
March 14, 2003

5. Define $f : \mathbf{R}^3 \times \mathbf{R}^2 \to \mathbf{R}^2$ by

$$f(x,y,z,u,v) = (xu^2 + yzv + x^2z - 3,\ xyv^3 + 2zu - u^2v^2 - 2).$$

The Jacobian is given by

$$Jf(x,y,z,u,v) = \begin{bmatrix} u^2+2xz & zv & yv+x^2 & 2xu & yz \\ yv^3 & xv^3 & 2u & 2z-2v^2 & 3xyv^2-2u^2v \end{bmatrix}$$

which means that $Jf(1,1,1,1)$ is the block matrix $[M\ N]$, where

$$M = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix} \qquad N = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}.$$

We have $\det(N) = 2$, which means that the implicit function theorem applies, so we obtain a differentiable function $h : U \subset \mathbf{R}^3 \to \mathbf{R}^2$ where U is an open set containing $(1,1,1)$, such that $h(1,1,1) = (1,1)$ and $f(x,y,z,h_1(x,y,z),h_2(x,y,z)) \equiv 0$. In other words, $h(x,y,z) = (u,v)$. The Jacobian of h at $(1,1,1)$ is given by

$$Jh(1,1,1) = -N^{-1}M = -\frac{1}{2}\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & -2 \end{bmatrix}. \tag{1}$$

Note on this problem:

(1) The above formula (1) follows directly from the chain rule, and actually can be found in Fitzpatrick, although it doesn't look quite like that. Another way of looking at finding the Jacobian of h is by implicit differentiation: that is, we have u and v as functions of x, y, and z, so since

$$xu(x,y,z)^2 + yzv(x,y,z) + x^2z \equiv 3$$

we can take the x-derivative to find

$$u(x,y,z)^2 + 2xu(x,y,z)\frac{\partial u}{\partial x} + yz\frac{\partial v}{\partial x} + 2xz \equiv 0.$$

This is the rigorous version of what you probably learned in a high school calculus course. Taking the x, y, and z derivatives of both equations, we obtain six linear equations to solve for six unknowns (the partials of u and v). This basically ends up being what happens in (1).

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