If you don't understand anything about any of the solutions here, or if you spot mistakes, feel free to e-mail me at `zeyliger@fas.harvard.edu`.

Notation Note: These two problems were largely about using and understanding the notation for quotient spaces. We have two notations for equivalence classes now. $[u]$ (or $\overline{u}$) is the equivalence class of u modulo some equivalence relation. $u + V$ is the coset $\{u + v \mid v \in V\}$. In the case of quotient spaces, these are interchangeable and you can use either. $[u + V]$ doesn't mean anything in our context, so don't use that.

2

Let U be finite-dimensional, and V a subspace of U. We wish to show that $\dim(U/V) = \dim(U) - \dim(V)$.

Consider the map $L : U \to U/V$ given my $L(u) = u + V$. (See note 1.) L is surjective: given $v + U$ in U/V, $L(V) = v + U$. $\operatorname{Ker}(L) = V$:

$$\begin{aligned} &L(u) = \vec{0} \\ \Longrightarrow\ &u + V = 0 + V \\ \Longrightarrow\ &u \in V. \end{aligned}$$

Now, we can apply the First Isomorphism Theorem and the Rank-Nullity Theorem. We have $U \cong \Im(L)/Ker(L)$, and $\dim(\operatorname{Ker}(L)) + \dim(\Im L) = \dim(\Im(L)/\operatorname{Ker}(L))$. Hence, $\dim(V) + \dim(U/V) = \dim(U/V)$. We conclude,

$$\dim(U/V) - \dim(V) = \dim(U/V).$$

Notes:

1. You cannot say "consider the surjective map $L : U \to U/V$" without stating what the map actually is. Yes, we used this map in class, but you still need to define it.
2. It was possible to do this problem with a basis argument. When you claim something is a basis, be sure to check that the set spans and is linearly independent.

3

P_n is the vector space of polynomials (with real coefficients) of degree less than or equal to n and P_n^0 is the subspace of polynomials with terms of only even degree. We wish to find a basis for P_n/P_n^0.

We recall that the set

$$B_n = \left\{1, x, x^2, \ldots, x^n\right\}$$

is a basis for P_n. (We've gone over this example, so there's no need to do it out. If you do show that B is linearly independent, be sure your argument works. It's not LI because x can be anything; that argument fails in $\mathbb{F}_2$. Instead set $x = 0$ and factor.)

Similarly, the set

$$B_n^0 = \left\{1, x^2, x^4, \ldots, x^m \mid\right\},$$

where $m = n$ if n is even, and $m = n - 1$ if n is odd, is a basis for P_n^0.

Problem 2 above tells us that the dimension of P_n/P_n^0 is $n - m$. Consider the set

$$\mathcal{B} = \left\{x + B_n^0, x^3 + B_n^0, \ldots, x^{m-1} + B_n^0\right\}.$$

We must show $\mathcal{B}$ is linearly independent. Switching notation, take a linear combination of vectors in $\mathcal{B}$ such that not all the coefficients are zero,

$$c_1[x] + c_2[x^3] + c_3[x^5] + \cdots + c_{n-m}[x^{m-1}] = [0].$$

Picking a representative of the left and using the definition of our equivalence relation, this implies that

$$c_1 x + c_2 x^3 + \cdots + c_{n-m} x^{m-1} = 0,$$

which is a contradiction, because the x^i are linearly independent since B_n is a basis.

$\mathcal{B}$ is a set of linearly independent vectors contained in P_n/P_n^0 and has the same cardinality as the dimension of P_n/P_n^0 so we can conclude that $\mathcal{B}$ is a basis for P_n/P_n^0.

Notes:

1. It is easy to show that this set spans, but there's no reason to, given the previous problem.
2. I took off points for notation. The set $C = \left\{x, x^3, \ldots, x^m\right\}$ is not a basis for P_n/P_n^0. The elements of C are not even elements of P_n/P_n^0!