MATH 23A SOLUTION SET #6 (PART B)

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Problem (4). *Let $D : C^\infty \to C^\infty$ be the linear differential operator given by $D(f) = f' + af$, where a is some fixed real number. Find* all *eigenvalues and eigenvectors for this operator.*

Solution. We fix a real number λ and try to find an f which would be a λ-eigenvector for D. Such a function f must satisfy:

$$D(f) = f' + af = \lambda f \Rightarrow f' = (\lambda - a)f$$

so we solve the above linear differential equation as follows:

$$\frac{f'}{f} = \lambda - a$$

$$\Rightarrow \int \frac{f'(t)}{f(t)} dt = \int (\lambda - a) dt$$

$$\Rightarrow ln(f(x)) + C_1 = \int \frac{d(f(t))}{f(t)} = (\lambda - a)x + C_2$$

$$\Rightarrow f(x) = e^{(\lambda - a)x + C_2 - C_1} = Ke^{(\lambda - a)x}$$

Notice that the constant $K = e^{C_2 - C_1}$ above is obtained by *remembering the constant of integration when dealing with indefinite integrals*. Notice also that the set of solutions of the above equation can be written as $Span(e^{(\lambda - a)x})$, which is a vector space. Thus, if we forget the constant of integration, we get just one λ-eigenvector as a solution, namely $f(x) = e^{(\lambda - a)x}$, instead of the whole λ-eigenspace. Note that all solutions are C^∞ functions, and that no solution equals the zero function, except the one obtained for $K = 0$. Since the above procedure works for all λ, we conclude that all real numbers λ are eigenvalues of D (i.e. $spec(D) = \mathbb{R}$), and for any $\lambda \in \mathbb{R}$, the λ-eigenspace V_λ is given by:

$$V_\lambda = \{f(x) = Ke^{(\lambda - a)x} : K \in \mathbb{R}\}$$

Some of you considered special cases $\lambda = 0$ and $\lambda = a$ separately. This is fine, but notice there is no need for this, since both cases have been accounted for above. When $\lambda = 0$, we just get that

$$ker(D) = V_0 = \{f(x) = Ke^{-ax} : K \in \mathbb{R}\}$$

so that D has a nontrivial kernel, and when $\lambda = a$, we get that:

$$V_a = \{f(x) = K : K \in \mathbb{R}\}$$

so that all constant functions are a-eigenvectors for D. □

Problem (5). *Suppose λ is a non-zero eigenvalue for the linear transformation $A : V \to V$.*

1. *Show that λ^2 is an eigenvalue for A^2*
2. *If A is invertible, show that λ^{-1} is an eigenvalue for A^{-1}.*

Solution. (1) Since λ is an eigenvalue for A, we know that $\exists \vec{v} \in V$ such that $\vec{v} \neq \vec{0}$ and $A(\vec{v}) = \lambda \vec{v}$. For this $\vec{v}$, we get that

$$A^2(\vec{v}) = A(A(\vec{v})) = A(\lambda(\vec{v})) = \lambda A(\vec{v}) = \lambda(\lambda \vec{v}) = \lambda^2 \vec{v}$$

where we have simply used the linearity of A and the definition of eigenvectors. Thus, $\vec{v} \neq \vec{0}$ is an eigenvector of A^2 for the eigenvalue λ^2.

(2) Again, since λ is an eigenvalue for A, we know that $\exists \vec{v} \in V$ such that $\vec{v} \neq \vec{0}$ and $A(\vec{v}) = \lambda \vec{v}$. For an invertible linear transformation A, A^{-1} is a gain a linear transformation, and such that $AA^{-1} = A^{-1}A = I$. Thus, we have that

$$A(\vec{v}) = \lambda \vec{v} \Rightarrow A^{-1}(\lambda \vec{v}) = \vec{v}$$

But we know A^{-1} is linear, so we get:

$$\vec{v} = A^{-1}(\lambda \vec{v}) = \lambda A^{-1}(\vec{v})$$

and after multiplying each side by λ^{-1}, we conclude that

$$\lambda^{-1} \vec{v} = A^{-1}(\vec{v})$$

so that $\vec{v} \neq \vec{0}$ is an eigenvector of A^{-1} for the eigenvalue λ^{-1}. $\square$