Solution Set 6D

Math 23a
November 13, 2002

8. a) Define $A : \mathbf{R}^4 \to \mathbf{R}^4$ by $A(a,b,c,d) = (-b,a,-d,c)$. Then A is clearly a linear map; it has an inverse $A^{-1}(a,b,c,d) = (b,-a,d,-c)$. Suppose that $\lambda \in \mathbf{R}$ and $\mathbf{v} = (a,b,c,d) \in \mathbf{R}^4$ such that $A\mathbf{v} = \lambda\mathbf{v}$. Then

$$-b = \lambda a \qquad a = \lambda b \qquad -d = \lambda c \qquad c = \lambda d$$

and therefore

$$-b = \lambda^2 b \qquad -d = \lambda^2 d.$$

If $b \neq 0$ or $d \neq 0$ then $\lambda^2 = -1$, which is impossible. Thus $b = d = 0$ so by the above equations, $a = c = 0$, and thus $\mathbf{v} = 0$. Therefore A has no eigenvectors and so A has no (real) eigenvalues.

A heuristic reason why A has no eigenvalues is that A is simply a ninety degree rotation in two copies of the plane.

b) Define $A : \mathbf{R}^4 \to \mathbf{R}^4$ by $A(a,b,c,d) = (-b,a,c,d)$. It is clear that A is linear, and A has an inverse $A^{-1}(a,b,c,d) = (b,-a,c,d)$. Suppose that $\lambda \in \mathbf{R}$ and $\mathbf{v} = (a,b,c,d) \in \mathbf{R}^4$ such that $A\mathbf{v} = \lambda\mathbf{v}$. Then

$$-b = \lambda a \qquad a = \lambda b \qquad \Longrightarrow \qquad -b = \lambda^2 b$$

so if $b \neq 0$ then $\lambda^2 = -1$, which cannot happen. Therefore $b = 0$ so $a = 0$. We also have

$$c = \lambda c \qquad d = \lambda d$$

so if either is nonzero, then $\lambda = 1$. Conversely,

$$A(0,0,c,d) = 1 \cdot (0,0,c,d)$$

for any $c, d \in \mathbf{R}$ so 1 is indeed an eigenvalue. Therefore A has the unique eigenvalue 1. In order to show that the dimension of the 1-eigenspace $\dim V_1$ is less than four, it suffices to find a vector that is not an eigenvector (because if the dimension were four then $V_1 = \mathbf{R}^4$). We conclude by noting that $A(1,0,0,0) = (0,1,0,0)$ which is not a multiple of $(1,0,0,0)$.

c) If we define $A(a,b,c,d) = (a+b, b+b, c+b, d+b)$ then A is a linear map with inverse $A^{-1}(a,b,c,d) = (a - b/2, b/2, c - b/2, d - b/2)$. We have

$$A(1,0,0,0) = 1 \cdot (1,0,0,0)$$
$$A(1,1,1,1) = 2 \cdot (1,1,1,1)$$

so $(1,0,0,0)$ and $(1,1,1,1)$ are eigenvectors with distinct eigenvalues.

Note that since $\mathbf{e}_1 = (1,0,0,0)$ and $\mathbf{v} = (1,1,1,1)$ are linearly independent, we could have extended them to a basis $(\mathbf{e}_1, \mathbf{v}, \mathbf{v}_1, \mathbf{v}_2)$ and defined A to take arbitrary (linearly independent) values on each of these basis vectors (so in particular, we could set $A\mathbf{e}_1 = \lambda_1\mathbf{e}_1$ and $A\mathbf{v} = \lambda_2\mathbf{v}$), but this is not very constructive.

---

9. a) Suppose that $A = SBS^{-1}$, i.e. $AS = SBS^{-1}S = SB$. Let λ be an eigenvalue of B with (non-zero) eigenvector $\mathbf{v}$. Then

$$AS\mathbf{v} = SB\mathbf{v} = S(\lambda\mathbf{v}) = \lambda S\mathbf{v}$$

so since $S\mathbf{v} \neq 0$ (because $\mathbf{v} \neq 0$ and S is injective), we know that λ is also an eigenvalue of A with eigenvalue $S\mathbf{v}$. Therefore $\operatorname{spec} B \subset \operatorname{spec} A$.

Since $B = S^{-1}AS$, we can use the same argument (with B and A swapped and S and S^{-1} swapped) to conclude that if γ is an eigenvalue of A with eigenvector $\mathbf{w}$ then γ is also an eigenvalue of B with eigenvector $S^{-1}\mathbf{w}$. Thus $\operatorname{spec} A \subset \operatorname{spec} B$ so combining this with the previous paragraph, $\operatorname{spec} A = \operatorname{spec} B$.

b) From the previous part, we know that if λ is an eigenvalue of B with eigenvector $\mathbf{v}$ then $S\mathbf{v}$ is an eigenvector of A with the same eigenvalue, and conversely, if $S\mathbf{v}$ is an eigenvector of A then $S^{-1}S\mathbf{v} = \mathbf{v}$ is an eigenvector of B with the same eigenvalue. Therefore $\mathbf{v}$ is an eigenvector of B if and only if $S\mathbf{v}$ is an eigenvector of A (both with eigenvalue λ). Thus

$$V_{\lambda,A} = \{S\mathbf{v} \mid \mathbf{v} \in V_{\lambda,B}\} = SV_{\lambda,B}$$

and similarly,

$$V_{\lambda,B} = \{S^{-1}\mathbf{w} \mid \mathbf{w} \in V_{\lambda,A}\} = S^{-1}V_{\lambda,A}.$$

Since S is a bijective linear map, this means that S restricts to an isomorphism of $V_{\lambda,B}$ with $V_{\lambda,A}$.

Notes on these problems:

(1) Approximately 80% of the people who turned in this problem set divided by zero at some point. Remember that

$$\lambda^2 b = -b$$

only implies that $\lambda^2 = -1$ when $b \neq 0$. I.e. you can't say that $ax = bx \implies a = b$ unless $x \neq 0$. Also, note that $(a, b, c, d) \neq \mathbf{0}$ does not imply that $b \neq 0$ — for instance, $(1, 0, 1, 1) \neq \mathbf{0}$.

(2) In general, it is not true that similar matrices have the *same* eigenspaces. For instance, if $A, B : \mathbf{R}^2 \to \mathbf{R}^2$ are defined by $A(x, y) = (x, -y)$ and $B(x, y) = (-x, y)$ then A and B are similar (check this!), but clearly $V_{1,A} = \{(a, 0) \mid a \in \mathbf{R}\}$ and $V_{1,B} = \{(0, b) \mid b \in \mathbf{R}\}$. It is true that these eigenspaces are always isomorphic (as you showed), i.e. $V_{1,A} \cong V_{1,B}$, but it does *not* make sense to say $v \cong w$ if $v \in V_{1,A}$ and $w \in V_{1,B}$. Be careful not to confuse the concepts of two things being *isomorphic* and being *the same* — the

former means that those things behave in the same way, and that everything that is true about the one is also true about the other, but one still has to identify them.

Joe Rabinoff