

MATH 23a, FALL 2002
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
(Final Version) Homework Assignment #7
Due: November 15, 2002

1. Read Sections 12 and 13 from Chapter 3 and Sections 14 and 15 from Chapter 4 of Curtis.

For problems 2 and 3, we consider the notion of invariant subspaces.

Definition. Let $L : V \rightarrow V$ be a linear transformation. We say that the subspace $W \subset V$ is **invariant** under L provided that $L(\mathbf{w}) \in W$, for every $\mathbf{w} \in W$.

For example, any eigenspace of L is invariant under L . For a slightly more involved example, consider $L : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $L(x, y, z) = (-y, x, z)$, which is a 90° rotation in the xy -plane. Then L has two invariant subspaces:

- $V_1 = \text{span}\{(0, 0, 1)\}$ is the eigenspace we have already studied
- $W = \text{span}\{(1, 0, 0), (0, 1, 0)\}$ (= the xy -plane itself) is invariant under L because $L(a, b, 0) = (-b, a, 0) \in W$.

2. (A) Let P_n be the vector space of polynomials of degree less than or equal to n , with real coefficients. Let $D : P_n \rightarrow P_n$ be the usual differentiation operator. Find non-trivial subspaces of P_n of each possible dimension which are invariant under D . (Note that the eigenspaces are insufficient to describe the behavior of D on P_n , because there is only one one-dimensional eigenspace. In other words, P_n is not diagonalizable with respect to D for $n \neq 0$, and so we consider its invariant subspaces.)
3. (A) Let P_n be as above. Define the following subspaces:

$$P_n^0 = \{p(x) \in P_n \mid p(-x) = p(x), \forall x\}$$

$$P_n^1 = \{p(x) \in P_n \mid p(-x) = -p(x), \forall x\}$$

(Note that the elements in P_n^0 are known as *even* polynomials, and the elements of P_n^1 are known as *odd* polynomials.)

- (a) Show that P_n^0 and P_n^1 are not invariant under D but are invariant under D^2 .
- (b) Show that $P_n \cong P_n^0 \oplus P_n^1$. (If you use bases for the two subspaces for this argument, then you should show that they *are* bases.)

4. (B) We define a linear map $U : V \longrightarrow V$ to be an *involution* provided that $U^2 = I$, where I is the identity map. In the following, consider such an involution U :

- (a) Show that U is invertible.
- (b) Show that P is a projection if and only if $2P - I$ is an involution.
- (c) Show that 1 and -1 are the only possible eigenvalues of U .
- (d) If V_λ represents the eigenspace for the eigenvalue λ under the map U , show that $V \cong V_1 \oplus V_{-1}$. (Note that this shows that an involution is diagonalizable!)

5. (C) Let P_3 be the vector space of polynomials of degree less than or equal to 3, with real coefficients.

Let $\mathfrak{B}_1 = \{1, x, x^2, x^3\}$, $\mathfrak{B}_2 = \{1, 1 + x, 1 + x^2, 1 + x^3\}$, and $\mathfrak{B}_3 = \{1 + x, 1 - x, x^2 - x^3, x^2 + x^3\}$ be bases for P_3 . Let $D : P_3 \longrightarrow P_3$ be the usual differentiation operator, and let $I : P_3 \longrightarrow P_3$ be the identity.

- (a) For each of the three bases, write down the matrix for D with respect to that basis (with the basis in question considered as the basis for both the domain and range).
- (b) Write down the matrix for I where the domain has basis \mathfrak{B}_1 and the range has basis \mathfrak{B}_2 .

6. (D) Let $V = \mathbb{R}^2$ be two-dimensional Euclidean space, with its usual x - and y - coordinate axes. Consider the linear transformation $L_\alpha : V \longrightarrow V$ that performs a reflection about the line $y = \alpha x$.

- (a) Write the matrix for L_α with respect to the basis $\mathfrak{B} = \{\mathbf{e}_1, \mathbf{e}_2\}$. (Hint: Use elementary geometry to compute $L_\alpha(\mathbf{e}_1)$ and $L_\alpha(\mathbf{e}_2)$.)
- (b) Calculate the matrix for $L_\beta \circ L_\alpha$ (with respect to \mathfrak{B}) in two ways: by multiplying the matrices for L_β and L_α , and by determining the matrix for the resulting composed linear transformation directly.
- (c) Show that the composed linear transformation $L_\beta \circ L_\alpha$ is a rotation. By what angle are vectors in \mathbb{R}^2 rotated under this transformation?