MATH 23B, SOLUTION SET FOR PS 7, PART D

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If you don't understand anything about any of the solutions here, or if you spot mistakes, feel free to e-mail me.

Problem 8

(a) Let $A = \{x \in [0,1] \mid \text{the decimal extension of } x \text{ has no 8's}\}$.
Claim: The content of A is 0.
Proof: To show that the content of A is zero, we need to find, given $\epsilon > 0$, a finite collection of closed rectangles whose area is less than epsilon and whose union contains A.
Consider

$$C_1 = \{[0, .8], [.9, 1]\} \qquad v(C_1) = .9$$
$$C_2 = \{[0, .08], [.09, .18], [.19, .28], \dots, [.90, .98], [.99, 1]\} \qquad v(C_2) = .9^2$$

and so on.

At every step, we are taking away $\frac{1}{10}$ of the numbers. So, $v(C_n) = \left(\frac{9}{10}\right)^n$. As $n \to \infty$, $v(C_n) \to 0$, so we can always find n such that $v(C_n) < \epsilon$, which is what we want.

As for a collection of rectangles whose volume is arbitrarily close to 0 and whose union is a subset of A, that's easy: just choose the empty collection.

(b) Let $B = \{n \in \mathbb{N} \mid \text{the decimal extension of } n \text{ has no 8's}\}$.
Claim: $\sum_{n \in B} \frac{1}{n}$ converges.
Proof:

$$\begin{aligned}\sum_{n\in B} \frac{1}{n} &= \frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{7} + \frac{1}{9} + \frac{1}{10} + \dots + \frac{1}{99} + \dots \\ &< \underbrace{\frac{1}{1} + \dots + \frac{1}{1}}_{9-1} + \underbrace{\frac{1}{10} + \dots + \frac{1}{10}}_{90-9-9} + \dots \\ &< 9 + \frac{9^2}{10} + \frac{9^3}{100} + \dots = 90.\end{aligned}$$

We've compared our series to a converging geometric series, so we are done.

Problem 9

Very, very few people got this problem right. If you got 5/10 (ast most people did), you had the right idea, but it was either not well-defined or you didn't prove continuity or both. If you got 10/10, kudos! You should read this over anyway, since it's probably informative. If you got 8/10 or 9/10, you probably fudged showing $f'(x)$ is not differentiable.

One way of representing the Cantor set is using ternary (base 3) notation. $C_1 = [0,1] - (1/3, 2/3)$. In tenary, C_1 includes $(0.0a_2a_3a_4\dots)_3$ and $0.2a_2a_3a_4\dots$ for any a_i. $(.1)_3$ is represented by $(.02222\dots)_3$. We see that C_1 is all the ternary numbers which don't have a 1 in the first decimal place. Similarly, C_2 is the ternary numbers which don't have a 1 in the first or second decimal place. The Cantor set is those numbers that can be represented without any 1s at all.

Using this defintion, we can show $\frac{1}{4} \in C$. Using long division in ternary, we can divide 1 by $(11)_3 = 4$. It turns out that $\frac{1}{4} = (.02020202\ldots)_3$. Since this number has no ones in it, it's a member of the Cantor set. On the other hand, 1/4 is not of the form $p/3^n$, so it cannot be the endpoint of any of the open intervals we take out. (There might be confusion between the terms "boundary point" and "endpoint". A point $x \in S \subset X$ is a "boundary point" if it is both in the closure of S and the closure of $X - S$. It turns out that every point in the Cantor set is a boundary point: C is closed, so its closure is itself. On the other hand, the closure of $[0, 1] - C$ is just $[0, 1]$. When I say "endpoint," I refer to the endpoints of the open intervals that we take out when we create C_n.)

Many students defined f on the Cantor set by saying that if $x \in C$, it must be of the form $x = 0, x = 1, or x = p/3^n$, i.e. the endpoint of one of the intervals. There turn out to be other points that you have to worry about. In fact, the Cantor set is uncountable! (There exists a surjective map from $C \to [0, 1]$; it is implicit in the second solution that I present. Understanding that the Cantor set is uncountable was one of the keys to this problem; you're supposed to think it strange that a set of measure zero is uncountable.) Also, if you defined it this way, you were saying that the image I of f was a subset of the rationals. But, then the image of a continuous map on a connected set would be disconnected! (A disconnection is $[0, \pi - 3) \cup (\pi - 3, 1]$.)

First Solution. Now, let us continue and define $f(x)$. First, let $(a_1^1, b_1^1) = (1/3, 2/3)$, $(a_1^2, b_1^2) = (1/9, 2/9)$, $(a_2^2, b_2^2) = (7/9, 8/9)$, and so on. That is, (a_k^n, b_k^n) is the nth segment we take out while forming the kth Cantor set. Then define $f(x)$ on $[0, 1]-C$ as $f(x) = \frac{2k-1}{2^n}$ if $x \in (a_k^n, b_k^n) \subset [0, 1]-C$. If $x \in C$, let $\{a_i\}$ be a sequence such that every $a_i \notin C$ and $\lim_{n\to\infty} a_n = x$. Such a sequence always exists because the Cantor set has measure zero. Define $f(x) = \lim_{n\to\infty} f(a_n)$ for $x \in C$. This is well-defined because f is monotone increasing on $[0, 1] - C$ and because given $x \in C$, there are sequences of rationals which converge in x both strictly from below and strictly from above.

Now we need to check that all the conditions are satisfied.

First, f is continuous. When $x \notin C$, this is clear, because f is constant on points $x \notin C$. For points $x \in C$, we defined it to be continuous! The definition of f and continuity are totally compatible.

Next, $f(0) = 0$ and $f(1) = 1$ because we are looking at the limit of $\frac{2k-1}{2^n}$ where $k = 1$ or $k = 2^{n-1}$. (For example, we built a sequence approaching 0 by taking a point in the leftmost non-Cantor segment of C_1, the leftmost non-Cantor segment of C_2, and so on.) For $x \notin C$, $f'(x) = 0$ because $[0, 1] - C$ is open. That is, given x, we can take a neighborhood around x that is also not contained in C, and the function is constant on this interval.

On the other hand, if $x \in C$, then $f'(x)$ is not defined. This requires proof. Consider the value of f on two left endpoints of adjacent non-Cantor segments in C_n. For example, consider $f(1/3) - f(1/9) = \frac{1}{2} - \frac{1}{4} = \frac{1}{2^2}$ and the next step $f(1/9) - f(1/27) = \frac{1}{2^3}$. On the other hand, $1/3-1/9 = \frac{2}{3^2}$ and $1/9-1/27 = \frac{2}{3^3}$. What this goes to show is that the slope of the line segment between the flat regions goes like $C\left(\frac{3}{2}\right)^2$, so it goes to infinity. When we examine the derivative, $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$, we can take advantage of the fact that any point $x \in C$ can be approximated to arbitrary precision by a left endpoint of a non-Cantor segment and use the Triangle Inequality to show that the derivative goes to infinity, i.e. does not exist.

Finally, we showed in class that C has content zero.

Caveat: Many of you defined f_n that worked for C_n and then wanted to define $f(x) = \lim_{n\to\infty} f_n(x)$. The problem is that you don't know much about $f(x)$ from such a construction. For example, you do not know that f is continuous. As an example of continuity failing, consider the sequence of functions $f_n : [0, 1] \to \mathbb{R}$ such that $f_n(x) = x^n$. In the limit, this converges "pointwise" to the function

$$f(x) = \begin{cases} 0 & 0 \le x < 1, \\ 1 & x = 1. \end{cases}$$

Whereas each $f_n(x)$ is continuous, the limit, $f(x)$, is clearly discontinuous. You'll find more examples and definitions of pointwise and uniform convergence in any analysis text.

Second Solution. An alternate (and pretty clever) way to define $f(x)$ is to use the ternary representation. First, if $x \in C$, then $x = (0.a_1a_2a_3\ldots)_3$ where $a_i \neq 1$. Define $f(x) = (0.b_1b_2b_3\ldots)_2$ where $b_1 = a_1/2$. What we're doing is converting every 2 in the ternary representation to a 1. This produces a binary number which is clearly in the range $[0, 1]$. (So far, we've defined a surjective map $f : C \to [0, 1]$, which shows that C is uncountable.) On the other hand, if $x \notin C$, then $x = (0.a_1a_2\ldots a_n1a_{n+2}a_{n+3}\ldots)_3$, where $a_i \neq 0$ for all $i < n+1$ and $a_i \neq 0$ for some $i > n+1$. Define $f(x) = (0.b_1b_2\ldots b_n1)_2$, where again $b_i = a_i/2$. We've truncated the process after the first occurence of a 1.

Checking that this function satisfies the correct properties turns out to be a little easier. $f(0) = 0$ and $f(1) = 1$ follow from the definition. For continuity, note that any points close to another in ternary turn out close to another in binary as well. Specifically, if a and b differ in the nth ternimal(?) position, they differ by less than $1/3^{n-1}$. $f(a)$ and $f(b)$ differ by less than $1/2^{n-1}$.

Points not on the Cantor set have neighborhoods (adjust the numbers after the 1) such that f is constant. The derivative is zero on these neighborhoods. On the other hand, if a point is on the Cantor set, it is not differentiable. Let $x = 0.a_1a_2a_3\cdots \in C$. If it happens to terminate, fill it up with $a_i = 0$ for $i > n$ for some n. Let y_n be x with the nth digit changed from 0 to 2 or from 2 to 0. Then, $|x - y_n| = \frac{1}{3^n}$. On the other hand, $|f(x) - f(y_n)| = \frac{1}{2^n}$. Then, $f'(x) = \lim_{y_n \to x} \frac{f(y_n)-f(x)}{y_n - x} = \lim_{n\to\infty} \left(\frac{3}{2}\right)^n = \infty$. So, $f'(x)$ does not exist. The sequence y_n is "good enough" because we can always find a y_n arbitrarily close to x. Any neighborhood of a point in the Cantor set contains (many many) other points in the set.

As a sidenote, it turns out that, up to homeomorphism, the Cantor set is the only totally disconnected, perfect, compact metric space.

If you want to see perhaps clearer expositions about the Cantor set, check out:

- `http://mathworld.wolfram.com/CantorSet.html`
- `http://www.mathacademy.com/pr/prime/articles/cantset/`
- `http://personal.bgsu.edu/~carother/cantor/Cantor1.html`
- `http://www.wikipedia.org/wiki/Cantor_set`