If you don't understand anything about any of the solutions here, or if you spot mistakes, feel free to e-mail me at `zeyliger@fas.harvard.edu`.

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Let $U : V \to V$ be an involution, i.e. $U^2 = I$.

a. We have that $U \circ U = I$, which implies that $U^{-1} = U$. U has an inverse, and is therefore invertible.

Many of you showed that U is bijective. There's nothing *per se* wrong about this, but noting the inverse is sufficient. It wasn't enough, however, to show that U is injective, because injectivity does not imply bijectivity if V is an infinite-dimensional vector-space. (Recall the shift operator from an earlier problem set.)

b. We wish to show that $2P - I$ is an involution if and only if P is a projection.

($\Leftarrow$) Let $P^2 = P$. Then,

$$(2P - I)^2 = (2P - I)(2P - I) = (2P)(2P) - 2PI - I2P + II \quad = 4P^2 - 4P + I = 4P - 4P + I = I.$$

($\Rightarrow$) Let $(2P - I)^2 = I$. Then, by the above calculation,

$$(2P - I)^2 = 4P^2 - 4P + I = I.$$

This implies that $4P^2 = 4P$, and hence $P^2 = P$, as desired.

c. Let λ be an eigenvalue of U. Then there exists $v \neq 0$ such that $U^2 v = \lambda^2 v$ and yet $U^2 v = Iv = v$. Hence, $\lambda^2 = 1$, and we may conclude $\lambda \in \{1, -1\}$.

d. Let us assume that F is not a field of characteristic zero (i.e. $1 + 1 \neq 0$). I will denote 2^{-1} by $\frac{1}{2}$.

Let V_1 and V_{-1} be the eigenspaces associated with the eigenvalues 1 and -1. Suppose $v \in V_1 \cap V_{-1}$. Then $U(v) = v = -v$, which implies $v = 0$. Thus, $V_1 \cap V_{-1} = \{0\}$.

Note that $\frac{1}{2}(v + U(v)) \in V_1$ and $\frac{1}{2}(v - U(v)) \in V_{-1}$:

$$U(\frac{1}{2}(v + U(v))) = \frac{1}{2}(U(v) + v)$$
$$U(\frac{1}{2}(v - U(v))) = \frac{1}{2}(U(v) - v).$$

Also,

$$v = \frac{1}{2}(v + U(v)) + \frac{1}{2}(v - U(v).$$

Hence, $V \subset V_1 + V_{-1}$. Because V_1 and V_{-1} are subspaces of V, $V_1 + V_{-1} \subset V$, so

$$V = V_1 + V_{-1}.$$

Since $V_1 \cap V_{-1} = \{0\}$, $V_1 + V_{-1} = V_1 \oplus V_{-1}$. We may conclude

$$V \cong V_1 + V_{-1}.$$

Several of you tried to use a map $L : V_1 \oplus V_{-1} \to V$ and show it is bijective. If you define such a map, you have to show it is well-defined. It is much easier to devine a map $L : V \to V_1 \oplus V_{-1}$ and show that is injective and that $V \supset V_1 \oplus V_{-1}$.