Math 23a, 2002.
Solution Set 7, Question 6.
Joshua Reyes

Question 6. Let $V = \mathbb{R}^2$ be a two-dimensional Euclidean space, with its usual x- and y-coordinate axes. Consider the linear transformation $L_\alpha : V \to V$ that performs a reflection about the line $y = \alpha x$.

(a) Write the matrix for L_α with respect to the basis $\mathfrak{B} = \{e_1, e_2\}$. (Hint: Use elementary geometry to compute $L_\alpha(e_1)$ and $L_\alpha(e_2)$.

(b) Calculate the matrix for $L_\beta \circ L_\alpha$ (with respect to $\mathfrak{B}$) in two ways: by multiplying the matrices for L_β and L_α, and by determing the matrix for the resulting composed linear transformation directly.

(c) Show that the composed linear transformation $L_\beta \circ L_\alpha$ is a rotation. By what angle are the vectors in $\mathbb{R}^2$ rotated under this transformation?

Answer. For the first part I'm going to skip the hint and play around with eigenstuff. After all, they're mad useful. Draw out the reflection (I'd supply diagrams, but it's a little tricky). What happens to vectors on the line $y = \alpha x$ (or, in vector language, something in the span$(1 \quad \alpha)^t$)? That's right. Nothing. So we've found an eigenvector with eigenvalue 1. Cool. Now what about something that's perpendicular to the line? Precisely! (You're good at this.) It gets flipped by a factor of -1. So now we've got an eigenbasis $\mathfrak{E} = \{v_1, v_2\}$. Let's write out the matrix with respect to that.

$$[L_\alpha]_{\mathfrak{E}} = \begin{bmatrix} L_\alpha(v_1) & L_\alpha(v_2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Now we'll just do a change of basis to get back to $\mathfrak{B}$. This is where all of that similiar matrices stuff comes into play. It's not so bad to go from $\mathfrak{E}$ to $\mathfrak{B}$. Say you want to find such a matrix P that does the job, just construct the matrix directly.

$$[P] = \begin{bmatrix} P(e_1) & P(e_2) \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} 1 & -\alpha \\ \alpha & 1 \end{bmatrix}.$$

To go the other way, just use P^{-1}. Now we're ready to find $[L_\alpha]_{\mathfrak{B}}$. First use P^{-1} to switch vectors in $\mathfrak{B}$ to vectors written in terms of $\mathfrak{E}$. Then apply your transformation $[L_\alpha]_{\mathfrak{E}}$. Notice how important it is to be in the same basis. This is why matrix representations can get a little tricky. Now go back to your original basis by way of P. All in all your matrix should look like this

$$[L_\alpha]_{\mathfrak{B}} = P[L_\alpha]_{\mathfrak{E}}P^{-1} = \frac{1}{1+\alpha^2}\begin{bmatrix} 1 & -\alpha \\ \alpha & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & \alpha \\ -\alpha & 1 \end{bmatrix} = \frac{1}{1+\alpha^2}\begin{bmatrix} 1-\alpha^2 & 2\alpha \\ 2\alpha & \alpha^2-1 \end{bmatrix}.$$

Now that looks pretty ugly, and trying to multiply that with another, equally bad matrix might be tricky. So I'm going to use some sneaky trig substitutions to make it looks a little prettier before moving onto part (b).

It's good to realise that the columns of L_α form an orthonormal basis. Since $[1 \quad \alpha]/(1+\alpha^2)$ lies on the unit circle, we can pretend that $1/(1+\alpha^2) = \cos\theta$ and that $\alpha/(1+\alpha^2) = \sin\theta$ for appropriate θ (say, $\arctan a$). Then plugging into our matrix, we have

$$L_\alpha = \begin{bmatrix} \cos^2\theta - \sin^2\theta & 2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \sin^2\theta - \cos^2\theta \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}.$$

This is how you usually see a reflection matrix.

Part (b) just amounts to matrix multiplication.

$$L_\beta L_\alpha = \begin{bmatrix} \cos 2\varphi & \sin 2\varphi \\ \sin 2\varphi & -\cos 2\varphi \end{bmatrix}\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} = \begin{bmatrix} \cos 2(\varphi-\theta) & -\sin 2(\varphi-\theta) \\ \sin 2(\varphi-\theta) & \cos 2(\varphi-\theta) \end{bmatrix},$$

where $\varphi = \arctan\beta$.

This is a particularly good form for part (c). An arbitrary rotation of angle ψ sends $e_1 \mapsto [\cos\psi \quad \sin\psi]^t$ and $e_2 \mapsto [-\sin\psi \quad \cos\psi]^t$. Comparing matrices, we see that

$$\mathrm{rot}\psi = \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix} = \begin{bmatrix} \cos 2(\varphi-\theta) & -\sin 2(\varphi-\theta) \\ \sin 2(\varphi-\theta) & \cos 2(\varphi-\theta) \end{bmatrix} = L_\alpha L_\beta$$

is a rotation by $2(\varphi-\theta)$ radians.