MATH 23a, FALL 2002
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
(Final Version) Homework Assignment # 8
Due: November 22, 2002

1. Read Sections 14–16 from Chapters 4 and 5 of Curtis and Sections 10.1, 12.1, and 15.1 from Fitzpatrick.

2. (A) Let a and b be real numbers. Using P to denote the set of positive elements, we define "less than" formally by the statement:

$$a < b \quad \text{iff} \quad b - a \in P.$$

Use axioms P1–P3 for an ordered field and the definition of absolute value to show that:

- If $0 < a < b$, then $0 < a^2 < b^2$.
- If $a^2 < b^2$, then $|a| < |b|$.

(Note that we have used this fact implicitly, such as when taking square roots, to justify several results about norms and inner products.)

3. (A) Let V be a vector space with an inner product $\langle \cdot, \cdot \rangle$ and an associated norm $|| \cdot ||$. We say that a linear transformation $A : V \longrightarrow V$ is *norm-preserving* if $||A\mathbf{v}|| = ||\mathbf{v}||$ for every $\mathbf{v} \in V$ and *inner-product-preserving* if $\langle A\mathbf{u}, A\mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle$ for all $\mathbf{u}, \mathbf{v} \in V$.

   (a) Show that A is inner-product-preserving if and only if it is norm preserving. (Hint #1: One way is easy! Hint #2: Expand the identity $||A(\mathbf{u} + \mathbf{v})||^2 = ||\mathbf{u} + \mathbf{v}||^2$.)

   (b) Let $V = \mathbb{R}^2$ with the usual inner product. Find all norm-preserving linear transformations/matrices.

4. (B)

   (a) Consider the vector space $V = C[-1, 1]$ of real-valued continuous functions on the closed interval $[-1, 1]$ with inner product

$$\langle f(x), g(x) \rangle = \int_{-1}^{1} f(x)g(x)\, dx.$$

   Orthogonalize the set of functions $\{1, x, x^2, x^3, x^4\}$ with respect to this inner product.

(b) With $V = C[0,1]$ and inner product

$$\langle f(x), g(x) \rangle = \int_0^1 f(x)g(x)\, dx,$$

orthogonalize the same set of functions from part (a).

(c) Now let $V = C[-1,1]$ as in part (a), but define $\langle \cdot, \cdot \rangle$ as in part (b). Show that this bilinear form is *not* an inner product on V. (Which properties of an inner product *are* satisfied?)

5. (C) A vector space V is called a *normed linear space* if there exists a map $|| \cdot || : V \to \mathbb{R}$ called the **norm** satisfying:

   i. $||c \cdot \mathbf{v}|| = |c| \cdot ||\mathbf{v}||, \forall c \in \mathbb{R}, \forall \mathbf{v} \in V$

   ii. $||\mathbf{v}|| \geq 0, \forall \mathbf{v} \in V$, and $||\mathbf{v}|| = 0$ iff $\mathbf{v} = \mathbf{0}$.

   iii. $||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||, \forall \mathbf{u}, \mathbf{v} \in V$

   (e. g. in a Euclidean space, the map $||\mathbf{v}|| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$ *is a norm.)*

   Show that the function $|| \cdot || : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by $||(x_1, \ldots, x_n)|| = \max\{|x_1|, \ldots, |x_n|\}$ defines a norm on $\mathbb{R}^n$. When $n > 1$, show that there is no inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$ such that this norm is associated to the inner product.

6. (D) A set S is called a **metric space** if there exists a function $d : S \times S \longrightarrow \mathbb{R}$ called the *distance* such that:

   i. $d(x, y) = d(y, x), \forall x, y \in S$

   ii. $d(x, y) \geq 0, \forall x, y \in S$ and $d(x, y) = 0$ iff $x = y$.

   iii. $d(x, y) \leq d(x, z) + d(z, y), \forall x, y, z \in S$

   (a) Show that if S is a Euclidean space, then it is also a metric space with distance given by

   $$d(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}||.$$

   (b) Let S be any set, and define the *discrete metric* as follows:

   $$d(x, y) = \begin{cases} 0 & , \text{ if } x = y \\ 1 & , \text{ if } x \neq y \end{cases}$$

   Show that S is a metric space.

   (c) Suppose S is a metric space with distance d. Show that S is also a metric space with new distance given by:

   $$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

(d) Consider $S = \mathbb{R}^2$, with $\mathbf{v} = (a, b)$ and $\mathbf{w} = (c, e)$. We define the *Memphis metric* by $d(\mathbf{v}, \mathbf{v}) = 0$ for any $\mathbf{v}$, and for $\mathbf{v} \neq \mathbf{w}$,

$$d(\mathbf{v}, \mathbf{w}) = \sqrt{a^2 + b^2} + \sqrt{c^2 + e^2}.$$

Show that S is a metric space.

7. (C) Let $V = \mathbb{R}^3$, and let $W = \text{span}\{(1, 1, 0), (3, 4, 5)\}$. Find the vector in W closest to the vector $\mathbf{v} = (7, 6, 4)$.

8. (Not required) Read p. 146 from Curtis and p. 363 of Fitzpatrick for the definition of the *transpose* of a matrix. Check that their definitions are equivalent to the following:

Definition. If $A = [a_{ij}]$ is an $n \times m$ matrix then we define its **transpose** $A^t = [a_{ji}]$ to be the $m \times n$ matrix whose rows are the columns of A. That is,

$$\text{if } A = \begin{bmatrix} a_{11} & \cdot \quad \cdot \quad \cdot & a_{1m} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{n1} & \cdot \quad \cdot \quad \cdot & a_{nm} \end{bmatrix} \text{ then } A^t = \begin{bmatrix} a_{11} & \cdot \quad \cdot \quad \cdot & a_{n1} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{1m} & \cdot \quad \cdot \quad \cdot & a_{mn} \end{bmatrix}.$$

Note that this definition applies equally well to vectors, which may be considered as $n \times 1$ matrices. Finally, read the three most important theorems concerning transposed matrices, and convince yourself of their validity.

Theorem: If A and B are $n \times m$, then $(A + B)^t = A^t + B^t$.

Theorem: If A is $n \times m$ and B is $m \times k$, then $(AB)^t = B^t A^t$.