HW 8 Part B

This was a pretty hard set, with the median only at 8.5 out of 10 and the mean an 8.4 out of 10.

3. CLAIM: If $h : A \to \mathbb{R}$ is a function such that the closure of

$$C = \{x \in A | f(x) \neq 0\}$$

is a set of measure zero, then H is integrable over A, and $\int_A h = 0$.

Proof: Because $\overline{C}$ is compact with measure zero, it has content zero. Therefore, fixing $\epsilon > 0$, it follows that there exists a partition P of A such that the sum of the volumes of all of the rectangles in P which are not disjoint from C is less than $\epsilon / \max\{\sup\{|h(x)| \mid x \in A\}, 1\}$ then $L(h, P) > -\epsilon$ and $U(h, P) < \epsilon$, so $\int_A h = 0$.

Observe that $f - g$ and $g - f$ are both functions from A to $\mathbb{R}$ each of which has the property that the closure of the set on which it is nonzero has measure 0 (and therefore are both integrable over A with integral 0 over A by the claim). Therefore, by Problem 4 of the previous set, if f is integrable over A, then $f + (g - f) = g$ is integrable over A and $\int_A g = \int_A f + \int_A (g - f) = \int_A f + 0 = \int_A f$. Similarly, if g is integrable over A, then $g + (f - g) = f$ is integrable over A as well and $\int_A f = \int_A g + \int_A f - g = \int_A g + 0 = \int_A g$.

NOTE: Many people tried to say something like "If F is the set of discontinuities of f and G is the set of discontinuities of g then $G \subset F \cup B$". However, this is not true; as a counterexample, suppose $f : [0, 1] \to \mathbb{R}$ is the zero function and $g : [0, 1] \to \mathbb{R}$ is zero except when $1/x \in N$ in which case $g(x) = 1$. Then, $g(0) = f(0) = 0$ but because there exist reciprocals of positive integers arbitrarily close to zero, g is discontinuous at zero. Therefore, making this mistake cost one point. The correct statement is actually $G \subset F \cup \overline{B}$; however, I actually made people prove this (with epsilons and deltas) or lose half a point.

In addition, in the latter part of the question many people tried to say something like, "Because T has measure zero, the integral of any function over T is zero" for some set T. While this would be true when dealing with Lebesgue integrals, the statement is not true for the Riemann integrals used in this class (for example, the constant function sending each number to 1 is not integrable over $[0, 1] \cap \mathbb{Q}$ even though the set has measure 0); the acceptable version of the statement is, "Because T has CONTENT zero, the integral of any function over T is zero". Again, usage of the incorrect "measure zero" version cost one point.