MATH 23B, SOLUTION SET FOR PS 8, PART C

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If you don't understand anything about any of the solutions here, or if you spot mistakes, feel free to e-mail me.

Problem 8 Define A and B as stated in the problem.

(a) One definition of the boundary is $\partial B = \overline{B} \cap \overline{B^c}$. Since, $A \subset B$, $\overline{A} \subset \overline{B}$. Since A is the set of rationals in $(0,1)$, $\overline{A} = [0,1]$. Since B is the union of open sets, B is open and, by definition, B^c is closed. Hence $\overline{(B^c)} = B^c$. Therefore, $\partial B = [0,1] \cap B^c = [0,1] \setminus B$, as desired.

(b) The previous problem implies that $\partial B \cup B = [0,1]$, which has measure 1. Suppose ∂B has measure zero. The hypothesis provides that B has measure less than $\epsilon = 1/2$. Then, $B \cup \partial B$ also has measure less than ϵ, which is a contradiction.

(c) If $x \in \partial B$, then, given any $\epsilon > 0$, there exists $q \in B$ such that $q \in B_\epsilon(x)$. This is because the rationals are dense and every rational in $(0,1)$ is in B. This implies that $o(\chi, x) = 1$ since the characteristic function is 1 on q and 0 on x. Hence χ is discontinuous on all points in ∂B. Since ∂B is a set of non-zero measure, χ is not integrable.