Math 23b, 2003.
Solution Set 8, Question 5.
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Question 5. Let $S \subset \mathbb{R}^3$ be the (bounded) intersection of the two (unbounded) cylinders $x^2 + z^2 \leq 1$ and $y^2 + z^2 \leq 1$. Show that the volume of S is $\frac{16}{3}$.

Answer. So there are a number of ways to do this. I'm going to pick the one that requires the least amount of explanation since the spring makes me lazy. Since both x and y are functions of z, it sort of makes sense to write the integral in terms of just z whose ranges go through the crazy interval $[-1, 1]$. As for the other two, move the things around to get $-\sqrt{z^2-1} \leq x,\, y \leq \sqrt{z^2-1}$. There are our bounds!

$$\begin{aligned}
\int_{-1}^{1}\int_{-\sqrt{z^2-1}}^{\sqrt{z^2-1}}\int_{-\sqrt{z^2-1}}^{\sqrt{z^2-1}} dx\,dy\,dz &= \int_{-1}^{1}\int_{-\sqrt{z^2-1}}^{\sqrt{z^2-1}} \Big[\,x\,\Big]_{-\sqrt{z^2-1}}^{\sqrt{z^2-1}} dy\,dz \\
&= \int_{-1}^{1}\int_{-\sqrt{z^2-1}}^{\sqrt{z^2-1}} 2\sqrt{z^2-1}\,dy\,dz \\
&= \int_{-1}^{1} \Big[2y\sqrt{z^2-1}\Big]_{-\sqrt{z^2-1}}^{\sqrt{z^2-1}} dz \\
&= 4\int_{-1}^{1} \left(1-z^2\right)\,dz
\end{aligned}$$

A lot of you started where I just left off, and that's cool so long as you told me why. So just for completeness' sake:

$$4\int_{-1}^{1}\left(1-z^2\right)\,dz = 4\left[z - \frac{z^3}{3}\right]_{-1}^{1} = 4 - \frac{4}{3} + 4 - \frac{4}{3} = \frac{16}{3}.$$