Solution Set 8B

Math 23a
December 5, 2002

4. a) We apply the Graham-Schmidt procedure to $\{1, x, x^2, x^3, x^4\}$. During this calculation, $\mathbf{u}_i$ will be an orthogonal set and $\mathbf{e}_i$ will be the corresponding orthonormal set. Note that if $\mathbf{e} = \mathbf{u}/\|\mathbf{u}\|$ then $\langle \mathbf{v}, \mathbf{e}\rangle \mathbf{e} = \langle \mathbf{v}, \mathbf{u}\rangle \mathbf{u}/\|\mathbf{u}\|^2$. We will also use x to denote a monomial and y to denote a variable of integration.

$$
\begin{aligned}
\mathbf{u}_1 &= 1 \qquad \mathbf{e}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = \frac{1}{\sqrt{2}} \cdot 1 \\
\mathbf{u}_2 &= x - \frac{\langle x, \mathbf{u}_1\rangle}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 = x - \frac{1}{2}\int_{-1}^{1} y\, dy \cdot 1 = x \\
\|\mathbf{u}_2\|^2 &= \int_{-1}^{1} y^2\, dy = \left[\frac{1}{3}y^3\right]_{-1}^{1} = \frac{2}{3} \\
\mathbf{e}_2 &= \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \sqrt{\frac{3}{2}}x \\
\mathbf{u}_3 &= x^2 - \frac{\langle x^2, \mathbf{u}_2\rangle}{\|\mathbf{u}_2\|^2}\mathbf{u}_2 - \frac{\langle x^2, \mathbf{u}_1\rangle}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 = x^2 - \frac{3}{2}\int_{-1}^{1} y^3\, dy \cdot x - \frac{1}{2}\int_{-1}^{1} y^2\, dy \cdot 1 \\
&= x^2 - \frac{1}{2}\left[\frac{1}{3}y^3\right]_{-1}^{1} \cdot 1 = x^2 - \frac{1}{3} \cdot 1 \\
\|\mathbf{u}_3\|^2 &= \int_{-1}^{1}\left(y^2 - \frac{1}{3}\right)^2 dy = \int_{-1}^{1}\left(y^4 - \frac{2}{3}y^2 + \frac{1}{9}\right) dy = \left[\frac{1}{5}y^5 - \frac{2}{9}y^3 + \frac{1}{9}x\right]_{-1}^{1} = \frac{8}{45} \\
\mathbf{e}_3 &= \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} = \frac{3\sqrt{5}}{2\sqrt{2}}\left(x^2 - \frac{1}{3} \cdot 1\right) \\
\mathbf{u}_4 &= x^3 - \frac{\langle x^3, \mathbf{u}_3\rangle}{\|\mathbf{u}_3\|^2}\mathbf{u}_3 - \frac{\langle x^3, \mathbf{u}_2\rangle}{\|\mathbf{u}_2\|^2}\mathbf{u}_2 - \frac{\langle x^3, \mathbf{u}_1\rangle}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 \\
&= x^3 - \frac{45}{8}\int_{-1}^{1} y^3\left(y^2 - \frac{1}{3}\right) dy \cdot \left(x^2 - \frac{1}{3}\right) - \frac{3}{2}\int_{-1}^{1} y^4\, dy \cdot x - \frac{1}{2}\int_{-1}^{1} y^3\, dy \cdot 1 \\
&= x^3 - \frac{3}{5} \cdot x \\
\|\mathbf{u}_4\|^2 &= \int_{-1}^{1}\left(y^3 - \frac{3}{5}y\right)^2 dy = \frac{8}{175} \\
\mathbf{e}_4 &= \frac{\mathbf{u}_4}{\|\mathbf{u}_4\|} = \frac{5\sqrt{7}}{2\sqrt{2}}\left(x^3 - \frac{3}{5} \cdot x\right)
\end{aligned}
$$

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$$
\begin{aligned}
\mathbf{u}_5 &= x^4 - \frac{\langle x^4, \mathbf{u}_4 \rangle}{\|\mathbf{u}_4\|^2}\mathbf{u}_4 - \frac{\langle x^4, \mathbf{u}_3 \rangle}{\|\mathbf{u}_3\|^2}\mathbf{u}_3 - \frac{\langle x^4, \mathbf{u}_2 \rangle}{\|\mathbf{u}_2\|^2}\mathbf{u}_2 - \frac{\langle x^4, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 \\
&= x^4 - \frac{175}{8}\int_{-1}^{1} y^4 \left(y^3 - \frac{3}{5}y\right) dy \cdot \left(x^3 - \frac{3}{5}x\right) \\
&\quad - \frac{45}{8}\int_{-1}^{1} y^4 \left(y^2 - \frac{1}{3}\right) dy \cdot \left(x^2 - \frac{1}{3}\right) - \frac{3}{2}\int_{-1}^{1} y^5 \, dy \cdot x - \frac{1}{2}\int_{-1}^{1} y^4 \, dy \cdot 1 \\
&= x^4 - \frac{6}{7}\left(x^2 - \frac{1}{3}\right) - \frac{1}{5} = x^4 - x^2 + \frac{3}{35} \cdot 1 \\
\|\mathbf{u}_5\|^2 &= \int_{-1}^{1} \left(y^4 - y^2 + \frac{3}{35}\right)^2 dy = \int_{-1}^{1} \left(y^8 - 2y^6 + \frac{41}{35}y^4 - \frac{6}{35}y^2 + \frac{9}{1225}\right) dy \\
&= \frac{218}{11025} \\
\mathbf{e}_5 &= \frac{\mathbf{u}_5}{\|\mathbf{u}_5\|} = \frac{105}{\sqrt{218}}\left(x^4 - x^2 + \frac{3}{35} \cdot 1\right)
\end{aligned}
$$

So our orthonormal set is

$$
\left\{ \frac{1}{\sqrt{2}} \cdot 1, \sqrt{\frac{3}{2}}x, \frac{3\sqrt{5}}{2\sqrt{2}}\left(x^2 - \frac{1}{3} \cdot 1\right), \frac{5\sqrt{7}}{2\sqrt{2}}\left(x^3 - \frac{3}{5} \cdot x\right), \frac{105}{\sqrt{218}}\left(x^4 - x^2 + \frac{3}{35} \cdot 1\right) \right\}.
$$

b) Again we apply Graham-Schmidt.

$$
\begin{aligned}
\mathbf{u}_1 &= 1 \qquad \mathbf{e}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = 1 \\
\mathbf{u}_2 &= x - \frac{\langle x, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 = x - \int_0^1 y \, dy \cdot 1 = x - \frac{1}{2} \cdot 1 \\
\|\mathbf{u}_2\|^2 &= \int_0^1 \left(y - \frac{1}{2}\right)^2 dy = \int_0^1 \left(y^2 - y + \frac{1}{4}\right) dy = \frac{1}{12} \\
\mathbf{e}_2 &= \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = (3\sqrt{2})\left(x - \frac{1}{2} \cdot 1\right) \\
\mathbf{u}_3 &= x^2 - \frac{\langle x^2, \mathbf{u}_2 \rangle}{\|\mathbf{u}_2\|^2}\mathbf{u}_2 - \frac{\langle x^2, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 \\
&= x^2 - 12\int_0^1 y^2 \left(y - \frac{1}{2}\right) dy \cdot \left(x - \frac{1}{2} \cdot 1\right) - \int_0^1 y^2 \, dy \cdot 1 \\
&= x^2 - \left(x - \frac{1}{2} \cdot 1\right) - \frac{1}{3} \cdot 1 = x^2 - x + \frac{1}{6} \cdot 1 \\
\|\mathbf{u}_3\|^2 &= \int_0^1 \left(y^2 - y + \frac{1}{6}\right)^2 dy = \frac{1}{180} \\
\mathbf{e}_3 &= \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} = 6\sqrt{5}\left(x^2 - x + \frac{1}{6} \cdot 1\right)
\end{aligned}
$$

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$$
\begin{aligned}
\mathbf{u}_4 &= x^3 - \frac{\langle x^3, \mathbf{u}_3\rangle}{\|\mathbf{u}_3\|^2}\mathbf{u}_3 - \frac{\langle x^3, \mathbf{u}_2\rangle}{\|\mathbf{u}_2\|^2}\mathbf{u}_2 - \frac{\langle x^3, \mathbf{u}_1\rangle}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 \\
&= x^3 - 180\int_0^1 y^3\left(y^2 - y + \frac{1}{6}\right)\,dy \cdot \left(x^2 - x + \frac{1}{6}\cdot 1\right) \\
&\quad - 12\int_0^1 y^3\left(y - \frac{1}{2}\right)\,dy \cdot \left(x - \frac{1}{2}\cdot 1\right) - \int_0^1 y^3\,dy \cdot 1 \\
&= x^3 - \frac{3}{2}\left(x^2 - x + \frac{1}{6}\cdot 1\right) - \frac{9}{10}\left(x - \frac{1}{2}\cdot 1\right) - \frac{1}{4}\cdot 1 \\
&= x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}\cdot 1 \\
\|\mathbf{u}_4\|^2 &= \int_0^1 \left(y^3 - \frac{3}{2}y^2 + \frac{3}{5}y - \frac{1}{20}\right)^2\,dy = \frac{1}{2800} \\
\mathbf{e}_4 &= \frac{\mathbf{u}_4}{\|\mathbf{u}_4\|} = 20\sqrt{7}\left(x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}\cdot 1\right) \\
\mathbf{u}_5 &= x^4 - \frac{\langle x^4, \mathbf{u}_4\rangle}{\|\mathbf{u}_4\|^2}\mathbf{u}_4 - \frac{\langle x^4, \mathbf{u}_3\rangle}{\|\mathbf{u}_3\|^2}\mathbf{u}_3 - \frac{\langle x^4, \mathbf{u}_2\rangle}{\|\mathbf{u}_2\|^2}\mathbf{u}_2 - \frac{\langle x^4, \mathbf{u}_1\rangle}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 \\
&= x^4 - 2800\int_0^1 y^4\left(y^3 - \frac{3}{2}y^2 + \frac{3}{5}y - \frac{1}{20}\right)\,dy \cdot \left(x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}\cdot 1\right) \\
&\quad - 180\int_0^1 y^4\left(y^2 - y + \frac{1}{6}\right)\,dy \cdot \left(x^2 - x + \frac{1}{6}\cdot 1\right) \\
&\quad - 12\int_0^1 y^4\left(y - \frac{1}{2}\right)\,dy \cdot \left(x - \frac{1}{2}\cdot 1\right) - \int_0^1 y^4\,dy \cdot 1 \\
&= x^4 - 2\left(x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}\cdot 1\right) - \frac{12}{7}\left(x^2 - x + \frac{1}{6}\cdot 1\right) \\
&\quad - \frac{4}{5}\left(x - \frac{1}{2}\cdot 1\right) - \frac{1}{5}\cdot 1 \\
&= x^4 - 2x^3 + \frac{9}{7}x^2 - \frac{2}{7}x + \frac{1}{70}\cdot 1 \\
\|\mathbf{u}_5\|^2 &= \int_0^1 \left(y^4 - 2y^3 + \frac{9}{7}y^2 - \frac{2}{7}y + \frac{1}{70}\right)^2\,dy = \frac{1}{44100} \\
\mathbf{e}_5 &= \frac{\mathbf{u}_5}{\|\mathbf{u}_5\|} = 210\left(x^4 - 2x^3 + \frac{9}{7}x^2 - \frac{2}{7}x + \frac{1}{70}\cdot 1\right)
\end{aligned}
$$

So our orthonormal set is

$$
\begin{aligned}
\Bigg\{&1, (3\sqrt{2})\left(x - \frac{1}{2}\cdot 1\right), 6\sqrt{5}\left(x^2 - x + \frac{1}{6}\cdot 1\right), \\
&20\sqrt{7}\left(x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}\cdot 1\right), \\
&210\left(x^4 - 2x^3 + \frac{9}{7}x^2 - \frac{2}{7}x + \frac{1}{70}\cdot 1\right)\Bigg\}.
\end{aligned}
$$

c) If we define f by

$$
f(x) = \begin{cases} x & \text{if } x < 0 \\ 0 & \text{if } x \geq 0 \end{cases}
$$

then $f \in C[-1,1]$ but $\langle f, f \rangle = 0$, which contradicts positive-definiteness. Thus $\langle \cdot, \cdot \rangle$ is not an inner product. It is easy to see, however (indeed, the same proof holds as for the actual inner products), that $\langle \cdot, \cdot \rangle$ is bilinear and symmetric.

Notes on this problem:

(1) There is no note one.

(2) For part (c), the important thing is to give an example of a function that contradicts positive-definiteness; you can't simply note that one might exist. If you drew a picture, since it's so obvious what will happen, that's fine; however, in general a "proof by pictures" is not a proof, and you should write down a function explicitly.

(3) In case you didn't notice, I didn't check every step to make sure your algebra was okay, and if your answer looked right, or just didn't look too wrong, that's fine. What's important is that you demonstrate that you understand how to do Graham-Schmidt. See note 5.

(4) Many people have the definition of positive-definiteness mixed up. To review: a form $\langle \cdot, \cdot \rangle$ is said to be *positive-definite* if:

- $\langle \mathbf{u}, \mathbf{u} \rangle > 0$ when $\mathbf{u} \neq 0$ and
- $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ when $\mathbf{u} = 0$.

I'm not sure why so many people thought the definition was $\langle \mathbf{u}, \mathbf{v} \rangle = 0 \iff \mathbf{u} = \mathbf{v}$ — in parts (a) and (b), where you have bona-fide inner products, you found many many pairs of non-equal orthogonal vectors.

(5) I highly recommend the use of calculators for problems like this. I use Maple myself; Mathematica is also good if you can stomach the licensing hassles. These programs are *very useful* for any mathematician, pure or applied, and you should understand how to use some computer algebra system. The point is, we assume you can do simple integrals and prime factorizations and other things that you'd know from BC calc, so feel free to seek computerized aid. The point of the problem sets is that you demonstrate to us that you can do what we *don't* assume you know (i.e. Graham-Schmidt). A slight warning: there will probably be some hairy integrals next semester that you'll have to do out though (although Mathematica might give you a hint as to what the answer is).

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