If you don't understand anything about any of the solutions here, or if you spot mistakes, feel free to e-mail me.

5

$\|v = (x_1, x_2, \ldots, x_n)\| = \max\{|x_1|, |x_2|, \ldots, |x_n|\}$. We check that this is a norm:

a.

$$\begin{aligned}\|c \cdot v\| &= \max\{|cx_1|, |cx_2|, \ldots, |cx_n|\} \\ &= \max\{c|x_1|, c|x_2|, \ldots, c|x_n|\} \\ &= c\max\{|x_1|, |x_2|, \ldots, |x_n|\} \\ &= c\|v\|.\end{aligned}$$

b. If $v \neq 0$, then some component x_i is non-zero, so $\|v\| \geq |x_i| > 0$. If $v = 0$, then $\|v\| = 0$.

c. Note that $|x_i + y_i| \leq |x_i| + |y_i|$ by the triangle inequality. Let $\|v\| = a$, $\|w\| = b$. Then,

$$\begin{aligned}\|v + w\| &= \max\{|x_1 + y_1|, |x_2 + y_2|, \ldots\} \\ &\leq \max\{|x_1| + |y_1|, |x_2| + |y_2|, \ldots\} \\ &\leq \max\{a + b, a + b, \ldots\} \\ &= \|v\| + \|w\|.\end{aligned}$$

Now, assume $\|v\| = \sqrt{\langle v, v\rangle}$. First, we use bilinearity to establish the parallelogram identity:

$$\begin{aligned}\langle v + w, v + w\rangle + \langle v - w, v - w\rangle &= 2\langle v, v\rangle + 2\langle w, w\rangle \\ \Rightarrow \|v + w\|^2 + \|v - w\|^2 &= 2(\|v\|^2 + \|w\|^2)\end{aligned}$$

Let $v = (1, 0, 0, \ldots, 0)$ and $w = (0, 1, 0, \ldots, 0)$. (Note that these vectors always exist when $\dim V > 1$.) Then, $\|v + w\| = 1 = \|v - w\|$ and $\|v\| = \|w\| = 1$. Therefore, by the identity above, $2 = 4$, which is a contradiction.

Note that I assumed only the bare minimum—$\|v\|$ comes from an inner product. Then I used the properties of an inner product to find some contradiction. Inner products don't have to have some "form", they just have to satisfy a few identities.

Also, **not all inner products are dot products**! Grahm-Shmidt is a method of constructing a basis in which an inner product *is* a dot product, so be wary of what basis you're in. If you're in an orthonormal basis, then dot-product away. If not, watch out!

Also, I saw several proofs that did not use the box norm at all. If you're not using one of the assumptions of the problem, chances are you're missing something—there are in fact norms which come from inner products.

7

I'm going to try to be as efficient as possible, in that I'm not going to solve a single linear system.

First, to find $W^\perp$, we use the cross product. (If you don't know this trick, you should learn it—it's really handy.)

$$\begin{aligned}(1,1,0)\times(3,4,5) &= (i+j)(3i+4j+5k)\\ &= 5i-5j+1k=(5,-5,1).\end{aligned}$$

So, $W^\perp = \operatorname{span}\{(5,-5,1)\}$. (I used the relations $ij=k$, $jk=i$, $ki=j$.) (You could also do this by solving a linear system. Convince yourself you've done the same thing.)

Then, we can project $v=(7,6,4)$ onto the line spanned by $w=(5,-5,1)$:

$$\begin{aligned}\text{proj}_w &= (w\cdot v)\frac{w}{\langle w,w\rangle}\\ &= (\frac{15}{17},\frac{-15}{17},\frac{3}{17}).\end{aligned}$$

Finally, we can subtract off this projection from v to get

$$v-(\frac{15}{17},\frac{-15}{17},\frac{3}{17}) = \boxed{(\frac{104}{17},\frac{117}{17},\frac{65}{17})}.$$