MATH 23b, SPRING 2003
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Homework Assignment # 9
Due: April 25, 2003

Homework Assignment #9 (Final Version)

1. Read Fitzpatrick, Section 18.4 and the beginning of Chapter 19.

2. (A) Spherical coordinates in n dimensions

   (a) Three-dimensional Euclidean space can be represented via the standard spherical coordinates transformation:

   $$(x, y, z) = T(\rho, \theta, \varphi) = (\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho \cos\varphi),$$

   where ρ is radius of the sphere on which (x, y, z) lies (the distance from the point to the origin), θ is the angle $(x, y, 0)$ makes with the x-axis (the longitude), and φ is the angle (x, y, z) makes with the z-axis (the latitude).
   Compute $|\det JT|$ as a function of ρ, θ, and φ.

   (b) More generally, n-dimensional spherical coordinates are given by the map $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ given as:

   $$\begin{aligned} x_1 &= \rho \cos\varphi_1 \\ x_2 &= \rho \sin\varphi_1 \cos\varphi_2 \\ x_3 &= \rho \sin\varphi_1 \sin\varphi_2 \cos\varphi_3 \\ &\vdots \\ x_{n-1} &= \rho \sin\varphi_1 \cdots \sin\varphi_{n-2} \cos\theta \\ x_n &= \rho \sin\varphi_1 \cdots \sin\varphi_{n-2} \sin\theta \end{aligned}$$

   Show by induction that

   $$|\det JT| = \rho^{n-1} \sin^{n-2}\varphi_1 \sin^{n-3}\varphi_2 \cdots \sin^2\varphi_{n-3} \sin\varphi_{n-2}.$$

   (c) Let $B^4 = \{(x, y, z, w) | x^2 + y^2 + z^2 + w^2 \leq 1\}$ be the unit ball in $\mathbb{R}^4$, and let $f(x, y, z, w) = e^{(x^2+y^2+z^2+w^2)^2}$. Use the spherical coordinates change of variables to compute the integral $\int_{B^4} f$. (Sorry, but we did B^3 in class.)

3. (B) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be positive and continuous, and suppose that

   $$\int\int_D f = \int_0^1 \left(\int_y^{\sqrt{2-y^2}} f(x, y)\, dx \right) dy.$$

   Sketch the region D and interchange the order of integration.

4. (C) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be continuous, and let B_ε be the ball of radius ε centered at the point $\mathbf{x} \in \mathbb{R}^n$. Show that:

$$\lim_{\varepsilon \to 0} \frac{1}{v(B_\varepsilon)} \int_{B_\varepsilon} f = f(\mathbf{x}).$$

5. (D) Use the change of variables $u = x - y$ and $v = x + y$ to evaluate the integral $\int\int_D e^{(x-y)/(x+y)}\, dx\, dy$, where D is the region in $\mathbb{R}^2$ bounded by the axes $x = 0$ and $y = 0$ and the line $x + y = 1$.

6. (E) Show that the function $f(x) = \frac{\sin x}{x}$ is integrable (in the sense defined below) but not absolutely integrable on $(0, \infty)$ in the following steps:

   (a) If we define $\displaystyle\int_0^\infty f(x)\, dx = \lim_{k\to\infty} \int_0^k f(x)\, dx$, then $f(x) = \frac{\sin x}{x}$ is integrable on $(0, +\infty)$, as we show in three steps:

      i. Show that $\displaystyle\int_0^\infty e^{-xy}\, dy = \frac{1}{x}$, if $x > 0$.

      ii. Use integration by parts to show that $\displaystyle\int_0^\infty e^{-xy} \sin x\, dx = \frac{1}{1+y^2}$, if $y > 0$.

      iii. Use (and explain/justify the use of) Fubini's Theorem to show that $\displaystyle\int_0^\infty \frac{\sin x}{x}\, dx = \int_0^\infty \int_0^\infty e^{-xy} \sin x\, dx\, dy = \int_0^\infty \frac{1}{1+y^2}\, dy.$ (What is the value of this last integral?)

   (b) Show that $f(x) = \frac{\sin x}{x}$ is not absolutely integrable in the following steps:

      i. Show that $\displaystyle\int_{2k\pi}^{(2k+1)\pi} \frac{\sin x}{x}\, dx \geq \frac{2}{(2k+1)\pi}.$

      ii. Given a compact subset $A \subset (0, +\infty)$, choose $m \in \mathbb{N}$ such that $A \subset [0, 2m\pi]$. For each $n > m$, define

$$B_n = [0, 2m\pi] \cup \bigcup_{k=m}^{n} [2k\pi, (2k+1)\pi].$$

      Using part (a) and the fact that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2k+1}$ diverges (why?), show that:

$$\lim_{n\to\infty} \int_{B_n} \frac{\sin x}{x}\, dx = \infty.$$

      (Why does this show that f is not absolutely integrable on $(0, +\infty)$?)

7. (Moral Homework)

Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be given by $f(x, y) = \dfrac{1}{(x^2 + y^2)^2}$, and consider the region $B = \{(x, y) \in \mathbb{R}^2 \mid x, y \geq 0 \text{ and } 0 < x^2 + y^2 \leq 1\}$.

(Note that f is unbounded on this region.)

Construct an approximating sequence of compact sets and determine whether or not f is integrable on B. *(Hint: Use polar coordinates.)*