MATH 23B, SOLUTION SET FOR PS 8, PART C

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If you don't understand anything about any of the solutions here, or if you spot mistakes, feel free to e-mail me.

Problem 2

(a) We've set $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, and $z = \rho\cos\phi$. Then,

$$JT = \begin{pmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -r\sin\phi \end{pmatrix}.$$

We calculate

$$\begin{aligned} |\det(JT)| &= \left|-\rho^2\sin^3\phi\cos^2\theta - \rho^2\sin\phi\cos^2\phi\sin^2\theta - \rho^2\sin^3\phi\sin^2\theta - \rho^2\sin\phi\cos^2\phi\cos^2\theta\right| \\ &= \left|-\rho^2\sin\phi(\sin^2\phi+\cos^2\phi)(\sin^2\theta+\cos^2\theta)\right| \\ &= \left|-\rho^2\sin\phi\right| = \rho^2\sin\phi, \end{aligned}$$

when $\phi \in [0,\pi]$.

(b) The $n = 3$ base case is shown above. We take x_i to be defined as in the problem set. Let us denote by ∇x_i the gradient in T_n, the transformation with n variables. Now, if we consider T_{n+1}, we get:

$$JT_{n+1} = \begin{pmatrix} ---\nabla x_1 --- & 0 \\ ---\nabla x_2 --- & 0 \\ \vdots & \vdots \\ ---\nabla x_1 --- & 0 \\ ---\cos\theta\nabla x_n --- & -\rho\sin\phi_1\sin\phi_2\ldots\sin\phi_{n-1}\sin\theta \\ ---\sin\theta\nabla x_n --- & -\rho\sin\phi_1\sin\phi_2\ldots\sin\phi_{n-1}\cos\theta. \end{pmatrix}$$

To take the determinant, we expand by the last column and take out a sin or cos by multilinearity to find

$$\begin{aligned} \det JT_{n+1} &= -(-1)^{2(n+1)-2}\rho\sin\phi_1\sin p_2\ldots\sin p_{n-1}\sin^2\theta\det T_{n-1} \\ &\quad + (-1)^{2(n+1)-1} - r\sin\phi_1\sin\phi_2\ldots\sin\phi_{n-1}\cos^2\theta \\ &= \pm\rho\sin p_1\sin p_2\ldots\sin p_{n-1}\rho^{n-2}\sin^{n-3}\phi_1\sin^{n-4}\phi_2\ldots\sin\phi_{n-3} \\ &= \pm\rho^{n-1}\sin^{n-2}\phi_1\sin^{n-3}\phi_2\ldots\sin\phi_{n-2}, \end{aligned}$$

and the absolute value of this is the desired result.

(c) Let $T(\rho,\phi_1,\phi_2,\theta) = (x,y,z,w) = (\rho\cos\phi_1, \rho\sin\phi_1\cos\phi_2, \rho\sin\phi_1\sin\phi_2, \cos\theta, \rho\sin\phi_1\sin\phi_2\sin\theta)$. Then,

$$\int_{B^4} f = \int_0^\pi\int_0^\pi\int_0^{2\pi}\int_0^1 e^{\rho^4}\rho^3\sin^2\phi_1\sin\phi_2 d\rho d\theta d\phi_1 d\phi_2$$

Letting $u = \rho^4$ and $du = 4\rho^3 d\rho$,

$$
\begin{aligned}
&= \int_0^{\pi} \int_0^{\pi} \int_0^{2\pi} \int_0^{1} \frac{e}{4} \sin^2 \phi_1 \sin \phi_2 du d\theta d\phi_1 d\phi_2 \\
&= \int_0^{\pi} \int_0^{\pi} \int_0^{2\pi} \frac{e-1}{4} \sin^2 \phi_1 \sin \phi_2 d\theta d\phi_1 d\phi_2 \\
&= \int_0^{\pi} \int_0^{\pi} \frac{\pi(e-1)}{4} \sin^2 \phi_1 \sin \phi_2 d\phi_1 d\phi_2 \\
&= \frac{\pi(e-1)}{4} \int_0^{\pi} \int_0^{\pi} \frac{1-\cos 2\phi_1}{2} \sin \phi_2 d\phi_1 d\phi_2 \\
&= \frac{\pi(e-1)}{4} \int_0^{\pi} \left(\frac{\pi}{2} - 0\right) \sin \phi_2 d\phi_2 \\
&= \frac{\pi^2(e-1)}{2}.
\end{aligned}
$$