

HW 9 Part C

Most of you did really well on this set; the mean was a 9.6/10 and the median was a perfect 10/10.

4. Let $g : \mathbb{R}^n \rightarrow \mathbb{R}$ be defined as follows: if $y \in \mathbb{R}^n$ then $g(y) = f(y) - f(x)$. Then g is a continuous function with $g(x) = 0$. Picking an $\eta > 0$ it follows that there exists some $\delta > 0$ such that $|g(y)| < \eta$ whenever $|y| < \delta$. Now define the functions $g_\eta : \mathbb{R}^n \rightarrow \mathbb{R}$ and $h_\eta : \mathbb{R}^n \rightarrow \mathbb{R}$ as follows: $g_\eta(y) = g(y)$ if $y \in B_\delta$ and 0 otherwise, and $h_\eta(y) = g(y)/\eta$. Therefore,

$$\frac{1}{v(B_\delta)} \int_{B_\delta} g = \frac{1}{v(B_\delta)} \int_{[-\delta, \delta] \times [-\delta, \delta]} g_\eta = \frac{\eta}{v(B_\delta)} \int_{[-\delta, \delta] \times [-\delta, \delta]} h_\eta.$$

As the last integrand takes on values of magnitude at most 1, it follows that

$$\left| \frac{1}{v(B_\delta)} \int_{B_\delta} g \right| \leq \frac{\eta}{v(B_\delta)} \cdot v(B_\delta) = \eta$$

(as the volume of a region is defined as the integral of 1 over that region), and therefore, as ϵ approaches 0, $1/v(B_\epsilon) \int_{B_\epsilon} g = 0$. As $f(y) = g(y) + g(x)$ for all $x \in X$, it follows that if $\delta > 0$, $1/v(B_\delta) \int_{B_\delta} f$ is equal to the sum of $1/v(B_\delta) \int_{B_\delta} g$ and $1/v(B_\delta) \int_{B_\delta} k$ where k is a constant equal to $f(x)$. As the first of these terms approaches zero as δ approaches zero and the second of these terms is equal to $k/v(B_\delta) \int_{B_\delta} 1 = (k/v(B_\delta)) \cdot v(B_\delta) = k$, it follows that the limit of $1/v(B_\epsilon) \int_{B_\epsilon} f$ as ϵ approaches 0 is equal to $f(x)$.

NOTE: The vast majority of people used a different method to solve this problem; they bounded $\int_{B_\epsilon} f$ by $\int_{B_\epsilon} m(f, x, \epsilon)$ and $\int_{B_\epsilon} M(f, x, \epsilon)$ (by monotonicity) and observed that those two integrals were equal to $v(B_\epsilon)m(f, x, \epsilon)$ and $v(B_\epsilon)M(f, x, \epsilon)$, respectively (by linearity). Both monotonicity and linearity obviously extend to generalized regions (and Jordan domains). However, many people (like last time) used the 'fact' that the integral of any function over a region with measure zero is 0 (when in reality you need content zero to do this); just like last time, usage of this 'fact' cost you one point.

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