Solution Set 9D

Math 23b
May 2, 2003

5. If we change variables to $u = x - y$ and $v = x + y$, then the transformation T which takes (u, v) to (x, y) is given by $T(u, v) = ((v + u)/2, (v - u)/2)$, which has Jacobian $JT = \left[\begin{smallmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{smallmatrix}\right]$. Thus $\det JT = 1/2$. Our original region D was given by the three conditions $x \geq 0, y \geq 0$, and $x+y \leq 1$. Substituting (u, v) for (x, y), the first condition becomes $u \geq -v$, the second condition becomes $u \leq v$, and the third becomes $v \leq 1$. By drawing those three lines, we find that $0 \leq v \leq 1$, so our integral becomes

$$\begin{aligned}
\int_0^1 \int_{-v}^{v} e^{u/v} \cdot \frac{1}{2}\, du\, dv &= \frac{1}{2} \int_0^1 [ve^{u/v}]_{u=-v}^{v}\, dv \\
&= \frac{1}{2} \int_0^1 (v(e - e^{-1}))\, dv \\
&= \frac{1}{2} \left[\frac{1}{2} v^2 (e - e^{-1}) \right]_0^1 \\
&= \frac{1}{4}(e - e^{-1}).
\end{aligned}$$

Notes on this problem:

(1) Technically, $f(x, y) = e^{(x-y)/(x+y)}$ is not defined at $(x, y) = (0, 0)$. However, the integral will not be affected if we simply declare that $f(0, 0) = 0$ — then f is a bounded almost-everywhere-continuous function on D.

(2) Remember that you have to take the Jacobian of the transformation taking (u, v) to (x, y), and not the reverse! This is analogous to the single-variable case: if you want to substitute $u = \chi(x)$ in the integral $\int f(x)\, dx$, then you need to find the inverse function $x = \tau(u)$ so that you can compute $dx = \tau'(u)\, du$ and substitute this into $\int f(\tau(u))\tau'(u)\, du$.

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