Math 23b, Spring 2003

Problem Set 9, Part E
Solutions written by Tseno Tselkov

Problem 6:

Proof.

(a) i. $$\int_0^\infty e^{-xy}dy = \lim_{i\to\infty}\int_0^i e^{-xy}dy = \lim_{i\to\infty}\left[-\frac{1}{x}e^{-xy}\right]_0^i =$$

$$= \lim_{i\to\infty}\left(-\frac{1}{x}e^{-ix}+\frac{1}{x}\right) = \frac{1}{x}$$

Notice that we used $x > 0$ to find the last limit.

ii. Using integration by parts we find that

$$\int e^{-xy}\sin x dx = -e^{-xy}\cos x - \int ye^{-xy}\cos x dx =$$

$$= -e^{-xy} - y\left(e^{-xy}\sin x + \int ye^{-xy}\sin x dx\right),$$

which implies that

$$(1+y^2)\int e^{-xy}\sin x dx = -e^{-xy}\cos x - ye^{-xy}\sin x.$$

$$\Rightarrow \int_0^\infty e^{-xy}\sin x dx = \lim_{i\to\infty}\left[\frac{-e^{-xy}\cos x - ye^{-xy}\sin x}{1+y^2}\right]_0^i = \frac{1}{1+y^2}.$$

iii. We can apply Fubini's Theorem since the function $e^{-xy}\sin x$ is continuous. Changing the order of integration at the fourth step we get

$$\int_0^\infty \frac{\sin x}{x}dx = \int_0^\infty \frac{1}{x}\cdot\sin x dx = \int_0^\infty\left(\int_0^\infty e^{-xy}dy\right)\sin x dx =$$

$$= \int_0^\infty\int_0^\infty e^{-xy}\sin x dy dx = \int_0^\infty\int_0^\infty e^{-xy}\sin x dx dy =$$

$$= \int_0^\infty \frac{1}{1+y^2}dy = \lim_{i\to\infty}\left[\text{arctany}\right]_0^i = \frac{\pi}{2}.$$

(b) i. When $x \in [2k\pi, (2k+1)\pi]$, since $\sin x \geq 0$, we have $\dfrac{\sin x}{x} \geq \dfrac{\sin x}{(2k+1)\pi}$.

$$\Rightarrow \int_{2k\pi}^{(2k+1)\pi} \frac{\sin x}{x} dx \geq \int_{2k\pi}^{(2k+1)\pi} \frac{\sin x}{(2k+1)\pi} dx =$$

$$= \frac{1}{(2k+1)\pi} \left[-\cos x\right]_{2k\pi}^{(2k+1)\pi} = \frac{2}{(2k+1)\pi}.$$

ii. First note that

$$\sum_{k=1}^{\infty} \frac{1}{2k+1} \geq \sum_{k=1}^{\infty} \frac{1}{3k} = \frac{1}{3} \sum_{k=1}^{\infty} \frac{1}{k},$$

which is known to be divergent. Now we have

$$\int_{B_n} \frac{\sin x}{x} dx = \int_0^{2m\pi} \frac{\sin x}{x} dx + \sum_{k=m}^{n} \int_{2k\pi}^{(2k+1)\pi} \frac{\sin x}{x} dx =$$

$$= C + \sum_{k=m}^{n} \frac{2}{(2k+1)\pi} = C + \frac{2}{\pi} \sum_{k=m}^{n} \frac{1}{2k+1}.$$

Hence

$$\lim_{n\to\infty} \int_{B_n} \frac{\sin x}{x} dx = \infty,$$

which directly implies that f is not absolutely integrable. $\square$