MATH 23a, FALL 2002
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Lecture # 6, supplement

Definition of the Real Numbers, $\mathbb{R}$

Definition: $\mathbb{R} = \{\{a_n\}_{n=1}^{\infty} \,|\text{the sequence is Cauchy, and } \; a_n \in \mathbb{Q}, \forall n\} \,/ \sim$

$$\text{where } \{a_n\} \sim \{b_n\} \text{ if and only if } \lim_{n\to\infty}(a_n - b_n) = 0.$$

That is, the real numbers are equivalence classes of Cauchy sequences, where two such sequences are equivalent if their difference (term-by-term) converges to zero.

Definition: If $[\{a_n\}]$ and $[\{b_n\}]$ are two real numbers, then we define their sum and product as follows:

$$[\{a_n\}] + [\{b_n\}] = [\{a_n + b_n\}]$$

$$[\{a_n\}] \cdot [\{b_n\}] = [\{a_n \cdot b_n\}]$$

We demonstrate that this multiplication is well-defined:

Suppose $\{a_n\} \sim \{c_n\}$ and $\{b_n\} \sim \{d_n\}$. Then we must show that $\{a_n \cdot b_n\} \sim \{c_n \cdot d_n\}$, or in other words, that $\lim_{n\to\infty}(a_n \cdot b_n - c_n \cdot d_n) = 0$.

Given $\varepsilon > 0$, we need to show that $|a_n \cdot b_n - c_n \cdot d_n| < \varepsilon$ for large enough n. Observe that

$$\begin{aligned} |a_n \cdot b_n - c_n \cdot d_n| &= |a_n \cdot b_n - b_n \cdot c_n + b_n \cdot c_n - c_n \cdot d_n| \\ &\leq |a_n \cdot b_n - b_n \cdot c_n| + |b_n \cdot c_n - c_n \cdot d_n| \\ &= |b_n| \cdot |a_n - c_n| + |c_n| \cdot |b_n - d_n| \end{aligned}$$

where we have used the triangle inequality in the second step. Because of the two equivalences we are given, we may make the terms $|a_n - c_n|$ and $|b_n - d_n|$ as small as we like, but we need to be able to put some restriction on the size of $|b_n|$ and $|c_n|$. We do this as follows.

Since $\{b_n\}$ is a Cauchy sequence, we know that given any $\varepsilon > 0$, there is some $N_1 \in \mathbb{N}$ such that $|b_n - b_m| < \varepsilon$ for $n, m > N_1$. We translate this statement by removing the absolute values to say that

$$-\varepsilon < b_n - b_m < \varepsilon$$

or that

$$b_m - \varepsilon < b_n < b_m + \varepsilon$$

But since this is true for *any* n and m greater than N_1, in particular it is true for a *fixed* m greater than N_1. So we choose a fixed m_1. Then

$$b_{m_1} - \varepsilon < b_n < b_{m_1} + \varepsilon$$

which implies that

$$|b_n| < \max\{|b_{m_1} - \varepsilon|, |b_{m_1} + \varepsilon|\} = M_1$$

for every $n > N_1$, where we take the expression on the right to be the definition of M_1.

Now we have bounded the terms $|b_n|$. Since $\{c_n\}$ is also a Cauchy sequence, we may bound these terms similarly. Given $\varepsilon > 0$, there is some N_2 and hence some m_2 such that

$$|b_n| < \max\{|b_{m_2} - \varepsilon|, |b_{m_2} + \varepsilon|\} = M_2,$$

whenever $n > N_2$.

Finally, we are ready to use our ability to make the terms $|a_n - c_n|$ and $|b_n - d_n|$ as small as we like. In particular, given the same $\varepsilon > 0$, we choose N_3 such that $|a_n - c_n| < \frac{\varepsilon}{2M_1}$, whenever $n > N_3$, and we choose N_4 such that $|b_n - d_n| < \frac{\varepsilon}{2M_2}$, whenever $n > N_4$.

Combining all of our results, if $\varepsilon > 0$ is given, we let $N = \max\{N_1, N_2, N_3, N_4\}$. If $n > N$, then all of our work above applies, and we see that

$$\begin{array}{rcl} |a_n \cdot b_n - c_n \cdot d_n| & \leq & |b_n| \cdot |a_n - c_n| + |c_n| \cdot |b_n - d_n| \\ & < & M_1 \cdot \frac{\varepsilon}{2M_1} + M_2 \cdot \frac{\varepsilon}{2M_2} \\ & = & \varepsilon \end{array}$$

Hence $\lim_{n\to\infty} (a_n \cdot b_n - c_n \cdot d_n) = 0$, which establishes that multiplication is well-defined.