

MATH 23b, SPRING 2004
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Questions for the Final

Directions: Here are five questions to help you prepare for the final, which will take place at 2:15 P.M. on Friday, May 28 in Sever 113. At least one of these questions will be on the final, either verbatim or in some small variation.

You are encouraged to work with other members of the class on these problems or to discuss them with the instructor, but in the interests of fairness, please do not ask the course assistants about these problems.

1. Let $f : [a, b] \rightarrow \mathbb{R}$ be an *increasing* function, that is, if $x, y \in [a, b]$ and $x < y$, then $f(x) < f(y)$. Show that f is integrable on $[a, b]$.
2. Let $A \subset \mathbb{R}^n$ be a closed rectangle, and suppose $f : A \rightarrow \mathbb{R}$ and $g : A \rightarrow \mathbb{R}$ are both integrable. Show that if $f(\mathbf{x}) > g(\mathbf{x}), \forall \mathbf{x} \in A$, then $\int_A f > \int_A g$.

For the record, this was a true/false question on the second portion of the midterm. I had intended to add the hypothesis that both f and g were continuous, in which case the proof of this statement is straightforward (exercise?). Without the continuity hypothesis, the statement is still true, but the proof is more interesting.

3. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$. Recall the Laplacian of f is defined as:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

We define a function as *harmonic* if $\nabla^2 f = 0$ and *subharmonic* if $\nabla^2 f > 0$.

Let $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ be the closed unit disk. Show that if $f : D \rightarrow \mathbb{R}$ is subharmonic, then f does not have a maximum on the interior of D .

4. Surface Area

One application of integration that we did not discuss in class concerns the computation of surface areas. If $S \subset \mathbb{R}^3$ is a smooth surface with parametrization $\varphi : U \rightarrow S$ for some open set $U \subset \mathbb{R}^2$, then the surface area of S is given by:

$$\text{surface area}(S) = \int_U \sqrt{\det([J\varphi(\mathbf{u})]^t[J\varphi(\mathbf{u})])} \, d\mathbf{u},$$

where $J\varphi$ is the Jacobian of φ and A^t is the transpose of A .

For more on this topic, please read Edwards, Section 5.4, but in any case, a few remarks are in order:

1. We say that φ is smooth if $\det J\varphi(\mathbf{u}) \neq 0, \forall \mathbf{u} \in U$.
2. Note that the matrix whose determinant we are computing is 2×2 . Furthermore, we do not need an absolute value on the determinant because the matrix $A^t A$ always has positive determinant when A has linearly independent columns (even when A itself is not square!).
3. This result is the natural analogue of the formula for computing arclength. (Here our matrix is 2×2 since we are computing *area* instead of length.)

For this problem, let $R > r > 0$, and consider the torus T (a donut!) in \mathbb{R}^3 obtained by taking the circle of radius r in the xz -plane centered at $(R, 0, 0)$ and rotating it around the z -axis.

Consider the parametrization $\varphi : [0, 2\pi] \times [0, 2\pi] \rightarrow T$ given by:

$$\varphi(u, v) = ((R + r \cos u) \cos v, (R + r \cos u) \sin v, r \sin u)$$

- (a) Explain how φ parametrizes T . (*i.e.* What do u and v represent?)
- (b) Calculate $J\varphi(u, v)$.
- (c) Calculate $\det([J\varphi(u, v)]^t[J\varphi(u, v)])$.
- (d) Calculate $\text{surface area}(T)$ using an iterated double integral.

5. Stokes' Theorem

Let S be a parametrized surface in \mathbb{R}^3 . That is, suppose there exists some $D \subset \mathbb{R}^2$ that is open, connected, and simply-connected, with a boundary $C = \partial D$ that is a piece-wise smooth, positively-oriented curve, and a bijective function $\varphi : \overline{D} \rightarrow S$ that is continuous on \overline{D} and continuously differentiable (except possibly on a set of measure zero) on D . By convention, denote the independent variables as u and v .

Let $F : U \rightarrow \mathbb{R}^3$ be a vector field that is class C^1 and represented by the components $F(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z))$, where $U \subset \mathbb{R}^3$ is some open set containing \overline{D} .

Then Stokes' Theorem says that:

$$\int \int_S \text{curl } F = \int_C F$$

where, if

$$G = \text{curl } F = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right),$$

then with the parametrization above,

$$\begin{aligned} \int \int_S \text{curl } F &= \int \int_S G \cdot \mathbf{n} \, dS \\ &= \int \int_D G(\varphi(u, v)) \cdot \left(\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} \right) \, du \, dv, \end{aligned}$$

where \mathbf{n} is the normal vector to the surface S , and u and v are the two variables parametrizing D as a subset of \mathbb{R}^2 .

- (a) By direct computation, prove Stokes' Theorem in the case that S is a rectangle parallel to the yz -plane (and whose edges are parallel to the y and z axes), that is, one of the form

$$S = \{(x, y, z) \mid x = a, b \leq y \leq b', c \leq z \leq c'\},$$

for some constants $a, b, c \in \mathbb{R}$.

- (b) Verify Stokes' Theorem for the vector field $F(x, y, z) = (z, x, y)$ on the surface $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \text{ and } z > 0\}$, the upper-hemisphere of the sphere of radius one, centered at the origin.