MATH 23a, FALL 2001
Midterm (in-class portion)
SOLUTIONS
October 31, 2001

1. True or False

   - Every Cauchy sequence of rational numbers converges to a rational number.
     False. This is why we construct the real numbers.
   - If $L : V \longrightarrow W$ is a surjective linear map, then it is invertible.
     False. L needs to be bijective to be invertible. In the case presented, it is possible to have a non-trivial kernel.
   - A linear transformation may have infinitely many eigenvalues.
     True. Consider $D : C^\infty \longrightarrow C^\infty$, the usual differential operator. You showed in HW #2(a) that every real number is an eigenvalue for D.
   - If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset V$ is a set of linearly independent vectors, then any $\mathbf{v} \in V$ may be written as a linear combination of these vectors in a unique way.
     False. These vectors may not span V.
   - If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset V$ is a set of vectors such that none of them is a scalar multiple of any of the others, then the set is linearly independent.
     False. If $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$, for example, the vectors could satisfy the condition but not be linearly independent.

   For the next three, consider the following:

   Let K be the kernel of the linear transformation $L : V \longrightarrow V$.

   - K is a subspace of V.
     True. We proved this in class.
   - K is an eigenspace of L.
     False. But only because "trivial eigenspaces" are not considered to be eigenspaces. (No points were deducted for any answer given to this problem!)
   - If K is finite-dimensional, then V/K is finite-dimensional.
     False. If V is finite-dimensional, then yes, but otherwise, no.

2. Show that the linear transformation $A : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ given by

$$A(x, y, z) = (x + y, y, 2z)$$

is not diagonalizable by following these steps:

(a) Define what it means for a linear transformation $L : V \longrightarrow V$ to be *diagonalizable*. (Hint: You may state either the definition or the theorem which gives an equivalent definition.)

(b) Determine all eigenvalues of A.

(c) Give bases for all the corresponding eigenspaces.

(d) Conclude that A is not diagonalizable.

Solution:

(a) A linear transformation $L : V \longrightarrow V$ is *diagonalizable* if the sum of the dimensions of the eigenspaces of L equals the dimension of V, or in symbols:

$$\sum_{\lambda \in \mathrm{Spec}(L)} dim(V_\lambda) = dim(V),$$

where $\mathrm{Spec}(L)$ is the set of eigenvalues of L and V_λ is the eigenspace corresponding to λ.

The theorem says that this is equivalent to V having an eigenbasis with respect to L.

(b) Suppose $\lambda \in \mathbb{R}$ is an eigenvalue of A. Then $A(x, y, z) = \lambda(x, y, z)$. Writing out both of these expressions and equating components, we get the three equations:

$$\begin{aligned} (1-\lambda)x + y &= 0 \\ (1-\lambda)y &= 0 \\ (2-\lambda)z &= 0 \end{aligned}$$

If $\lambda = 1$, then we find $\mathbf{e}_1 = (1, 0, 0)$ satisfies these equations. If $\lambda = 2$, then we find $\mathbf{e}_3 = (0, 0, 1)$ satisfies these equations. If $\lambda \neq 1$ and $\lambda \neq 2$, then we conclude that $z = 0$ (third equation), $y = 0$ (second equation), and therefore $x = 0$ (first equation). Hence no other real number is an eigenvalue.

(c) If $\lambda = 1$, then the equations from part (b) reduce to: $y = 0$ and $z = 0$. With no restriction on x, we see that

$$V_1 = \{(x, 0, 0) | x \in \mathbb{R}\} = \text{span}\{\mathbf{e}_1\}$$

If $\lambda = 2$, then the equations from part (b) reduce to: $-x + y = 0$ and $y = 0$, and hence we also conclude that $x = 0$. With no restriction on z, we see that

$$V_2 = \{(0, 0, z) | z \in \mathbb{R}\} = \text{span}\{\mathbf{e}_3\}$$

(d) $dim(V_1) = 1$ and $dim(V_2) = 1$, but $dim(V) = 3$, so the diagonalizability definition fails because $1 + 1 \neq 3$.

3. Consider the linear transformation $B : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ given by

$$B(x, y, z, w) = (z - x, y - z, 0, x - y)$$

(a) Find a basis for $K = Ker(B)$, and show that it is a basis.

(b) Define the *quotient space* U/V, where V is a subspace of a vector space U.

(c) Find a basis for $\mathbb{R}^4/K$, and show that it is a basis.

Solution:

(a) Suppose $\mathbf{v} = (x, y, z, w) \in Ker(B)$. Then $B(\mathbf{v}) = (z - x, y - z, 0, x - y) = (0, 0, 0, 0)$. Matching up the components yields the three equations: $z = x$, $y = z$, and $x = y$. Hence, given any x, our choices of y and z are forced to be the same, but w may be chosen independently, and we get:

$$Ker(B) = \{(x, x, x, w) | x, w \in \mathbb{R}\} = \text{span}\{(1, 1, 1, 0), (0, 0, 0, 1)\}$$

It is clear that these two vectors are linearly independent because they are not scalar multiples of each other.

(b) By definition, $U/V = \{\mathbf{u} + V | \mathbf{u} \in U\}$, where $\mathbf{u} + V = \{\mathbf{u} + \mathbf{v} | \mathbf{v} \in V\}$ is a coset of V, and $\mathbf{u}_1 + V = \mathbf{u}_2 + V$ if and only if $\mathbf{u}_1 - \mathbf{u}_2 \in V$.

(c) We know that $dim(U/V) = dim(U) - dim(V)$ when U is finite-dimensional, so in this case, we have $dim(\mathbb{R}^4/K) = dim(\mathbb{R}^4) - dim(K) = 4 - 2 = 2$. Hence any basis of $\mathbb{R}^4/K$ will have two elements, and it will suffice to check either that they span the space or that they are linearly independent. For simplicity, since neither $\mathbf{e}_2$ nor $\mathbf{e}_3$ is in the kernel, we claim:

$$\mathbb{R}^4/K = \text{span}\{\mathbf{e}_2 + K, \mathbf{e}_3 + K\}$$

We show linear independence of these two vectors:
Suppose $a_2(\mathbf{e}_2 + K) + a_3(\mathbf{e}_3 + K) = \mathbf{0} + K$. Then $(a_2\mathbf{e}_2 + a_3\mathbf{e}_3) + K = \mathbf{0} + K$, or in other words, $a_2\mathbf{e}_2 + a_3\mathbf{e}_3 = (0, a_2, a_3, 0) \in K$. By our determination of K in part (a), if the first component is 0, then so are the second and third components, and so $a_2 = 0$ and $a_3 = 0$. Hence the two vectors are linearly independent.

4. If $d \in \mathbb{Z}$ is a non-square (that is, there is no integer c such that $c^2 = d$), then $\sqrt{d}$ is an irrational number, and we define

$$\mathbb{Q}(\sqrt{d}) = \{a + b\sqrt{d} | a, b \in \mathbb{Q}\},$$

where $\mathbb{Q}$ is the field of rational numbers. We also define addition and multiplication as follows:

$$(a + b\sqrt{d}) + (c + e\sqrt{d}) = (a + c) + (b + e)\sqrt{d}$$

$$(a + b\sqrt{d}) \cdot (c + e\sqrt{d}) = (ac + bed) + (ae + bc)\sqrt{d}$$

It may be checked that $\mathbb{Q}(\sqrt{d})$ is a vector space over the field $\mathbb{Q}$ by verifying the axioms. Note that scalar multiplication is given implicitly by the second formula above, where we identify the rational number (the scalar) $a \in \mathbb{Q}$ with the element $a + 0\sqrt{d} \in \mathbb{Q}(\sqrt{d})$. Moreover, according to the rules for addition and multiplication above, it may be verified that $\mathbb{Q}(\sqrt{d})$ is actually a field.

Finally, we define a concept closely related to that of linear map. If F_1 and F_2 are fields, then a *field homomorphism* between them is a map $\varphi : F_1 \longrightarrow F_2$ satisfying:

$$\varphi(x + y) = \varphi(x) + \varphi(y), \forall x, y \in F_1$$

$$\varphi(xy) = \varphi(x)\varphi(y), \forall x, y \in F_1$$

(a) Find the dimension of $\mathbb{Q}(\sqrt{d})$ over $\mathbb{Q}$ (as a vector space) by finding a basis.

(b) Show that $\mathbb{Q}(\sqrt{2}) \cong \mathbb{Q}(\sqrt{-1})$ as vector spaces by constructing a bijective linear map between them.

(c) Show that $\mathbb{Q}(\sqrt{2})$ satisfies Axiom M4 (Multiplicative Inverses) for a field by finding the multiplicative inverse for the element $a + b\sqrt{2}$.

(d) Show that for any field homomorphism, $\varphi : F_1 \longrightarrow F_2$, we must have $\varphi(1) = 1$.

(e) Show that the map you constructed in part (b) is not a field isomorphism (bijective homomorphism).

Solution:

(a) $\mathbb{Q}(\sqrt{d}) = \text{span}\{1, \sqrt{d}\}$, where we identify 1 with the element $1 + 0\sqrt{d}$ and the element $\sqrt{d}$ with $0 + 1\sqrt{d}$.
If $a + b\sqrt{d} \in \mathbb{Q}(\sqrt{d})$, then $a + b\sqrt{d} = a(1) + b(\sqrt{d})$, so these two vectors span $\mathbb{Q}(\sqrt{d})$. If $a + b\sqrt{d} = 0$, then $a = 0$ and $b = 0$, so these vectors are linearly independent. Hence, $dim(\mathbb{Q}(\sqrt{d}) = 2$.

(b) Define $L : \mathbb{Q}(\sqrt{2}) \longrightarrow \mathbb{Q}(\sqrt{-1})$ by the rule:

$$L(a + b\sqrt{2}) = a + b\sqrt{-1}$$

We check the requirements:

- <u>L is linear</u>:
$$\begin{aligned} L(\alpha_1(a + b\sqrt{2}) + \alpha_2(c + d\sqrt{2})) &= L((\alpha_1 a + \alpha_2 c) + (\alpha_1 b + \alpha_2 d)\sqrt{2}) \\ &= (\alpha_1 a + \alpha_2 c) + (\alpha_1 b + \alpha_2 d)\sqrt{-1} \\ &= \alpha_1(a + b\sqrt{-1}) + \alpha_2(c + d\sqrt{-1}) \end{aligned}$$
- <u>L is injective</u>:
Suppose $L(a + b\sqrt{2}) - L(c + d\sqrt{2}) = 0$. Then $L((a - c) + (b - d)\sqrt{2}) = (a - c) + (b - d)\sqrt{-1} = 0$, and hence $a = c$ and $b = d$.
- <u>L is surjective</u>:
If $a + b\sqrt{-1} \in \mathbb{Q}(\sqrt{-1})$, then take $a + b\sqrt{2} \in \mathbb{Q}(\sqrt{2})$, and clearly $L(a + b\sqrt{2}) = a + b\sqrt{-1}$, by the definition of L.

(c) Given $a + b\sqrt{2} \in \mathbb{Q}(\sqrt{2})$ with not both a and b equal to 0, we require $x+y\sqrt{2} \in \mathbb{Q}(\sqrt{2})$ such that $(a+b\sqrt{2})(x+y\sqrt{2}) = 1+0\sqrt{2}$. Multiplying on the left and matching coordinates, we get the system of two equations in two unknowns:

$$ay + bx = 0$$

$$ax + 2by = 1$$

which we must solve for x and y in terms of a and b. Doing so yields:

$$x = \frac{a}{a^2 - 2b^2} \text{ and } y = \frac{-b}{a^2 - 2b^2}$$

and so

$$(a + b\sqrt{2})^{-1} = \frac{1}{a^2 - 2b^2}(a - b\sqrt{2}).$$

It is worth noting that $a^2 - 2b^2 \neq 0$ for any rational numbers a and b.

(d) According to the second rule for a field homomorphism: $\varphi(1) \cdot \varphi(1) = \varphi(1 \cdot 1) = \varphi(1)$ and hence $\varphi(1) = 0$ or 1. In the first case, if $a \in F_1$ were any other element, then we would have

$$\varphi(a) = \varphi(a) \cdot \varphi(1) = \varphi(a) \cdot \varphi(1) = \varphi(a) \cdot 0 = 0,$$

and so φ would be the trivial homomorphism. (Since the phrasing of the question did not rule out the trivial case, there will be great leniency in the grading of this part.)

(e) Using the map L from part (b), we compute $X = L((a+b\sqrt{2})(x+y\sqrt{2}))$ in two ways. On the one hand, we have $X = L((ax+2by)+(ay + bx)\sqrt{2}) = (ax + 2by) + (ay + bx)\sqrt{-1}$. On the other hand, according to the second rule for a field homomorphism, we have $X = L(a + b\sqrt{2}) \cdot L(x + y\sqrt{2}) = (a + b\sqrt{-1}) \cdot (x + y\sqrt{-1}) = (ax - by) + (ay + bx)\sqrt{-1}$.

The only way the two expressions for X match is if $-by = 2by$, which is not true in general, and so the second rule fails.