

Last Name: _____

First Name: _____

MATH 23b, SPRING 2002
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Final Exam (in-class portion)
May 23, 2002

Directions:

You have three hours for this exam. No calculators, notes, books, etc. are allowed. Please answer on the pages provided. (There are blank pages included after problems 4,5, and 6 for extra work.) Show all work!

Problem	Points	points by part	Score
1	12	1 each	
2	9	3 each	
3	8	4/2/2	
4	9	3 each	
5	12	6/3/3	
6	12	2/4/4/2	
Total	62	62	

1. True or False

- T** or **F** If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear, then Jf is linear.
- T** or **F** The function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable at $\mathbf{a} \in \mathbb{R}^n$ if and only if all of its directional derivatives exist.
- T** or **F** If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is differentiable, then f is continuously differentiable.
- T** or **F** If $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is locally invertible at $\mathbf{a} \in \mathbb{R}^n$, then $\det Jf(\mathbf{a}) \neq 0$.
- T** or **F** The quadratic form $q(x, y) = x^2 + 2xy + y^2$ is positive-definite.
- T** or **F** If $A \subset \mathbb{R}^n$ has content 0, then A has measure 0.
- T** or **F** Let $A \subset \mathbb{R}^n$ be a closed rectangle, and let $f : A \rightarrow \mathbb{R}$ be bounded. If P is a partition of A and P' is a refinement of P , then $L(f, P) \leq L(f, P')$.
- T** or **F** Let $A \subset \mathbb{R}^n$ be bounded and suppose ∂A has measure zero. If $f : A \rightarrow \mathbb{R}^n$ is bounded, then f is integrable on A .
- T** or **F** Let $A \subset \mathbb{R}^n$ be an open ball, and let $f : A \rightarrow \mathbb{R}$ be in the class C^2 . Then ∇f is conservative.
- T** or **F** Let $U \subset \mathbb{R}^n$ be an open ball, and let $\mathbf{x}_1, \mathbf{x}_2 \in U$. Let C be a piece-wise smooth curve from \mathbf{x}_1 to \mathbf{x}_2 , and let $\gamma_1 : [a, b] \rightarrow C$ and $\gamma_2 : [c, d] \rightarrow C$ be two parametrizations of C . Then
$$\int_a^b F(\gamma_1(t)) \cdot \gamma_1'(t) dt = \int_c^d F(\gamma_2(t)) \cdot \gamma_2'(t) dt.$$
- T** or **F** Let $U \subset \mathbb{R}^n$ be an open ball, and let $\mathbf{x}_1, \mathbf{x}_2 \in U$. If C_1 and C_2 are two piece-wise smooth curves from \mathbf{x}_1 to \mathbf{x}_2 , then $\int_{C_1} F = \int_{C_2} F$.
- T** or **F** Let $A \subset \mathbb{R}^n$ be a compact set, and let $f_k : A \rightarrow \mathbb{R}$ be a sequence of continuous functions. If the sequence converges point-wise to $f : A \rightarrow \mathbb{R}$, then f is continuous.

2. True or False with Justification

For the following statements, provide a short justification if the statement is true and a counter-example if the statement is false.

- (a) The set $A \subset \mathbb{R}^n$ has measure zero if and only if A is countable.
- (b) Let $f : [0, 1] \rightarrow \mathbb{R}$ be bounded. If $B = \{x \in [0, 1] | f(x) \neq 0\}$ is countable, then f is integrable on $[0, 1]$.
- (c) If $A \subset \mathbb{R}^n$ is a closed rectangle and $f : A \rightarrow \mathbb{R}$ is bounded on A , then the *upper integral* of f on A , defined by $\mathcal{U} \int_A f = \inf \{U(f, P) | P \text{ is a partition of } A\}$, exists.

3. Green's Theorem

- (a) State Green's Theorem (including all hypotheses!) for a vector field $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$.
- (b) Explain the impact of Green's Theorem when F is conservative. (*i.e.* How does this relate to the Fundamental Theorem of Line Integrals?)
- (c) Explain why Green's Theorem does not apply to the vector field

$$F(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)$$

on a region containing the origin.

4. A double integral

- (a) State the Change of Variables Theorem for multiple (iterated) integrals.
- (b) Let $A = \{(x, y) \in \mathbb{R}^2 \mid x, y \geq 0 \text{ and } \frac{1}{2} \leq x^2 + y^2 \leq 1\}$, and let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be given by:

$$f(x, y) = \frac{1}{(x^2 + y^2)^2}.$$

Sketch A , and use polar coordinates and iterated integrals to evaluate $\int_A f$.

- (c) Let f be as in part (b), and consider the region

$$B = \{(x, y) \in \mathbb{R}^2 \mid x, y \geq 0 \text{ and } 0 < x^2 + y^2 \leq 1\}.$$

(Note that f is unbounded on this region.)

Construct an approximating sequence of compact sets and determine whether or not f is integrable on B .

(Hint: Stay in polar coordinates.)

5. Uniform Convergence and the Bessel function

In part (a) of this problem, we consider a convergent sequence of functions and find a condition for when the derivatives of those functions also converge. In parts (b) and (c), we apply this result to an important function in the theory of differential equations.

- (a) Supply brief justifications (names of theorems, definitions, etc.) for the numbered steps in the proof of the following theorem.

Theorem.

For each $n \in \mathbb{N}$, let $f_n : \mathbb{R} \rightarrow \mathbb{R}$ be a continuously differentiable function (class C^1), and suppose that the sequence $\{f_n\}$ converges uniformly to f on (a, b) . Suppose further that the sequence $\{f'_n\}$ converges uniformly to g on (a, b) . Then $g(x) = f'(x), \forall x \in (a, b)$.

(In other words, $\lim_{n \rightarrow \infty} f'_n(x) = f'(x), \forall x \in (a, b)$.)

Proof:

1. Each f'_n is continuous, _____
2. and hence, for each $x \in (a, b)$, we have _____
$$\int_a^x g(t) dt = \lim_{n \rightarrow \infty} \int_a^x f'_n(t) dt.$$

3. Now, $\int_a^x f'_n(t) dt = f_n(x) - f_n(a)$, _____
4. and hence $\int_a^x g(t) dt = f(x) - f(a)$. _____
It follows that
$$f(x) = \int_a^x g(t) dt + f(a), \forall x \in (a, b).$$
 (algebraic re-arrangement) _____
5. Now, g is continuous, _____
6. and hence f is differentiable, and _____
$$f'(x) = g(x).$$

(b) The *Bessel function of order zero* is defined as a power series:

$$J_0(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{2^{2k} (k!)^2}.$$

Because J_0 has infinite radius of convergence, the sequence of partial sums:

$$f_n(x) = \sum_{k=0}^n (-1)^k \frac{x^{2k}}{2^{2k} (k!)^2},$$

converges to J_0 point-wise for all $x \in \mathbb{R}$.

Use the definition of uniform convergence to show that $\{f_n\}$ converges to J_0 uniformly on $(-1, 1)$. (For the record, the result is true for any interval, but it is mildly more difficult to show.)

(c) Apply part (a) to show that J_0 satisfies the differential equation:

$$J_0'' + \frac{1}{x} J_0' + J_0 = 0.$$

You may assume that your work in part (b) applies to both J_0' and J_0'' as well.

6. Surface Integrals

One application of integration that we did not discuss in class concerns the computation of surface areas. If $S \subset \mathbb{R}^3$ is a smooth surface with parametrization $\gamma : U \rightarrow S$ for some open set $U \subset \mathbb{R}^2$, then the surface area of S is given by:

$$\text{surface area}(S) = \int_U \sqrt{\det([J\gamma(\mathbf{u})]^t[J\gamma(\mathbf{u})])} \, d\mathbf{u},$$

where $J\gamma$ is the Jacobian of γ and A^t is the transpose of A .

A few remarks are in order:

1. We say that γ is smooth if $\det J\gamma(\mathbf{u}) \neq 0, \forall \mathbf{u} \in U$.
2. Note that the matrix whose determinant we are computing is 2×2 . Furthermore, we do not need an absolute value on the determinant because the matrix $A^t A$ always has positive determinant when A has linearly independent columns (even when A itself is not square!).
3. This result is the natural analogue of the formula for computing arclength. (Here our matrix is 2×2 since we are computing *area* instead of length.)

For this problem, let $R > r > 0$, and consider the torus T (a donut!) in \mathbb{R}^3 obtained by taking the circle of radius r in the xz -plane centered at $(R, 0, 0)$ and rotating it around the z -axis.

Consider the parametrization $\gamma : [0, 2\pi] \times [0, 2\pi] \rightarrow T$ given by:

$$\gamma(u, v) = ((R + r \cos u) \cos v, (R + r \cos u) \sin v, r \sin u)$$

- (a) Explain how γ parametrizes T . (*i.e.* What do u and v represent?)
- (b) Calculate $J\gamma(u, v)$.
- (c) Calculate $\det([J\gamma(u, v)]^t[J\gamma(u, v)])$.
- (d) Calculate $\text{surface area}(T)$ using an iterated double integral.