MATH 23b, SPRING 2004
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Homework Assignment # 10
Due: April 30, 2004

Homework Assignment # 10 (Final Version)

1. Read Edwards Sections 4.4–4.5.

2. (A) Let $S \subset \mathbb{R}^3$ be the (bounded) intersection of the two (unbounded) cylinders $x^2 + z^2 \leq 1$ and $y^2 + z^2 \leq 1$. Show that the volume of S is $\frac{16}{3}$.

   You can view an interactive feature with this object at:

   `http://www.math.umn.edu/~garrett/qy/Cylinders.html`

3. (A) Show that the volume of $B_1(0) \subset \mathbb{R}^4$ is $\frac{\pi^2}{2}$.

   (Hint: Use the fact that the volume of a ball in $\mathbb{R}^3$ of radius r is $\frac{4}{3}\pi r^3$.)

4. (*) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be positive and continuous, and suppose that

$$\int\int_D f = \int_0^1 \left(\int_y^{\sqrt{2-y^2}} f(x,y)\, dx \right) dy.$$

   Sketch the region D and interchange the order of integration.

5. (B) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be continuous, and let B_ε be the ball of radius ε centered at the point $\mathbf{x} \in \mathbb{R}^n$. Show that:

$$\lim_{\varepsilon \to 0} \frac{1}{v(B_\varepsilon)} \int_{B_\varepsilon} f = f(\mathbf{x}).$$

6. (C) In class, we considered the set $A = \mathbb{Q} \cap [0,1]$ and showed that it has measure zero. In particular, we showed that it was countable, that is, we could write it as $A = \{a_0, a_1, a_2, a_3 \ldots\}$. Given an $\varepsilon > 0$, we then covered it with rectangles $I_i = [a_i - \frac{\varepsilon}{2^{i+2}}, a_i + \frac{\varepsilon}{2^{i+2}}], \forall i \in \mathbb{N}$, so that

$$v\left(\bigcup_{i=0}^{\infty} I_i\right) \leq \sum_{i=0}^{\infty} v(I_i) = \varepsilon.$$

   Fix $\varepsilon = \frac{1}{2}$, and let $J_i = (a_i - \frac{1}{2^{i+3}}, a_i + \frac{1}{2^{i+3}})$ be the open rectangle equal to the interior of the corresponding I_i defined above.

   Let $B = \bigcup_{i=2}^{\infty} J_i$. *(Note that we are purposely omitting the first two sets, which cover 0 and 1, respectively, so that each $J_i \subset [0,1]$!)*

(a) Show that $\partial B = [0,1] \setminus B$.

(b) Show that ∂B does not have measure zero.

(c) Let χ_B be the characteristic function of B. Show that χ_B is not integrable on $[0,1]$.

(Note that although B is a "reasonable" set in the sense that it is the union of a countable collection of open sets, it does not have a "reasonable" boundary, and so χ_B is not integrable.)

7. (D) Spherical coordinates in n dimensions (from Edwards' problem 5.19):

(a) Three-dimensional Euclidean space can be represented via the standard spherical coordinates transformation:

$$(x,y,z) = T(\rho,\theta,\varphi) = (\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi),$$

where ρ is radius of the sphere on which (x,y,z) lies (the distance from the point to the origin), θ is the angle $(x,y,0)$ makes with the x-axis (the longitude), and φ is the angle (x,y,z) makes with the z-axis (the latitude).

Compute $|\det JT|$ as a function of ρ, θ, and φ.

(b) More generally, n-dimensional spherical coordinates are given by the map $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ given as:

$$\begin{aligned} x_1 &= \rho\cos\varphi_1 \\ x_2 &= \rho\sin\varphi_1\cos\varphi_2 \\ x_3 &= \rho\sin\varphi_1\sin\varphi_2\cos\varphi_3 \\ &\vdots \\ x_{n-1} &= \rho\sin\varphi_1\cdots\sin\varphi_{n-2}\cos\theta \\ x_n &= \rho\sin\varphi_1\cdots\sin\varphi_{n-2}\sin\theta \end{aligned}$$

Show by induction that

$$|\det JT| = \rho^{n-1}\sin^{n-2}\varphi_1\sin^{n-3}\varphi_2\cdots\sin^2\varphi_{n-3}\sin\varphi_{n-2}.$$

(c) Let $B^4 = \{(x,y,z,w) \mid x^2+y^2+z^2+w^2 \le 1\}$ be the unit ball in $\mathbb{R}^4$, and let $f(x,y,z,w) = e^{(x^2+y^2+z^2+w^2)^2}$. Use the spherical coordinates change of variables to compute the integral $\int_{B^4} f$.