MATH 23b, SPRING 2004
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
(Moral) Homework Assignment # 11
No due date

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1. Read Edwards, Chapter 5, especially Sections 5.1–5.2.

2. In the following problem, we compute the volume of the n-dimensional unit ball, $B^n \subset \mathbb{R}^n$, in the even and odd cases, respectively, to be:

$$v(B^{2m}) = \frac{\pi^m}{m!} \qquad \text{and} \qquad v(B^{2m+1}) = \frac{2^{m+1}\pi^m}{1 \cdot 3 \cdot 5 \cdots (2m+1)}$$

   (a) Read Edwards' problem 5.17 (p. 267) and problem 1.8 (p. 213) for one solution. Note that Edwards uses the notation I_n to denote the value of the integral $\displaystyle\int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta$.

   (b) Do Edwards' problem 5.18 (p. 267):
   Let $B^2 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\}$.
   Let $Q = \{(x_3, \ldots, x_n) \in \mathbb{R}^{n-2} \mid |x_i| \leq 1, \forall i\}$. Then $B^n \subset B^2 \times Q$.
   Let $\varphi : B^2 \times Q \to \mathbb{R}$ be the characteristic function of B^n. Then

$$v(B^n) = \int_{B^2} \left(\int_Q \varphi(x_1, \ldots, x_n) \, dx_3 \ldots dx_n \right) dx_1 \, dx_2.$$

   If $(x_1, x_2) \in B^2$ is fixed, then φ, considered as a function of the variables $x_3, \ldots, x_n$ is the characteristic function of B_r^{n-2}, the $(n-2)$-ball of radius $r = \sqrt{1 - x_1^2 - x_2^2}$. Hence

$$\int_Q \varphi(x_1, \ldots, x_n) \, dx_3 \ldots dx_n = (1 - x_1^2 - x_2^2)^{(n-2)/2} \cdot v(B^{n-2}).$$

   Now, introduce polar coordinates on $\mathbb{R}^2$ to show that

$$\int_{B^2} (1 - x_1^2 - x_2^2)^{(n-2)/2} \, dx_1 dx_2 = \frac{2\pi}{n}.$$

   Conclude that $v(B^n) = \frac{2\pi}{n} \cdot v(B^{n-2})$, and use induction and the base cases (that is, that $v(B^2) = \pi$ and $v(B^3) = \frac{4}{3}\pi$) to prove the given formulas for $v(B^n)$.

3. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be given by $f(x, y) = \dfrac{1}{(x^2 + y^2)^2}$, and consider the region $B = \{(x, y) \in \mathbb{R}^2 \mid x, y \geq 0 \text{ and } 0 < x^2 + y^2 \leq 1\}$.

   (Note that f is unbounded on this region.)

   Construct an *approximating sequence* of compact sets and determine whether or not f is integrable on B. *(Hint: Use polar coordinates.)*

4. Problem 5.7 from p. 264 of Edwards. Use the change of variables $u = x - y$ and $v = x + y$ to evaluate the integral $\int\int_D e^{(x-y)/(x+y)}\, dx\, dy$, where D is the region in $\mathbb{R}^2$ bounded by the axes $x = 0$ and $y = 0$ and the line $x + y = 1$.

5. Let $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be given by $F(x, y) = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2})$.

   Let C_0 be the unit circle in $\mathbb{R}^2$ parametrized by the function $\gamma_0 : [0, 1] \longrightarrow \mathbb{R}^2$, where $\gamma_0(t) = (\cos 2\pi t, \sin 2\pi t)$.

   Let C_1 be the circle of radius 1 centered at $(1, 1)$ in $\mathbb{R}^2$ parametrized by the function $\gamma_1 : [0, 1] \longrightarrow \mathbb{R}^2$, where $\gamma_1(t) = (1 + \cos 2\pi t, 1 + \sin 2\pi t)$.

   (a) Show that $\int_{C_0} F = 2\pi$.

   (b) Show that $\int_{C_1} F = 0$.

6. Let $F(x, y, z) = (z^3 + 2xy, x^2 + 1, 3xz^2)$ be a vector field on $\mathbb{R}^3$. Show that F is conservative by computing the partial derivatives of its component functions, and find $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ such that $F = \nabla f$.

7. Evaluate $\int_C 2xyz\, dx + x^2 z\, dy + x^2 y\, dz$, where C is a piece-wise smooth oriented curve from $(1, 1, 1)$ to $(1, 2, 4)$.