Solution Set 1

Math 23a
October 4, 2002

7. Let x be any positive real number. First we prove that x has a decimal expansion.

CLAIM. *There exists an integer k and integers $a_i \in \{0, 1, \ldots, 9\}$ with $a_k \neq 0$ such that*

$$x = \sum_{i=k}^{\infty} a_i \cdot 10^{-i}.$$

PROOF.

Let S be the set $\{n \in \mathbf{Z} \mid 10^n > x\}$. Since this set is bounded below by $\log_{10} x$, the well-ordering principle gives us a lower bound for S, i.e. a minimal integer K such that $x < 10^K$. Thus $10^K > x \geq 10^{K-1}$ (otherwise $K-1$ would be in S); we set $k = -(K-1)$.

The division algorithm states that we can find an integer a_k and a nonnegative real number x_{k+1} less than 10^{-k} such that

$$x = 10^{-k} a_k + x_{k+1}.$$

Now, if $a_k \geq 10$ then $x \geq 10^{-k+1}$, which is impossible; similarly, if $a_k \leq -1$ then $x < 0$ since $x_{k+1} < 10^{-k}$, which also can't happen. If $a_k = 0$ then $x = x_{k+1} < 10^{-k}$ which would contradict the fact that $x \geq 10^{-k}$. Thus $a_k \in \{1, 2, \ldots, 9\}$.

We will now inductively define all of the a_i for $i > k$. Let $n > k$ and suppose that there exist $a_k, a_{k+1}, \ldots, a_{n-1} \in \{0, 1, \ldots, 9\}$ such that

$$x = \sum_{i=k}^{n-1} a_i \cdot 10^{-i} + x_n$$

with $0 \leq x_n < 10^{-n+1}$. Note that we just showed that this is true in the base case $n = k$. We can apply a procedure similar to above to x_n — that is, we can find an integer a_n and real number x_{n+1} with $0 \leq x_{n+1} < 10^{-n}$ such that

$$x_n = 10^{-n} a_n + x_{n+1}.$$

As above, if $a_n \geq 10$ then $x_n \geq 10^{-k}$ which is impossible by the inductive hypothesis, and if $a_{k+1} \leq -1$ then $x_n < 0$ because $x_{n+1} < 10^{-n}$, which also can't happen. Thus $a_n \in \{0, 1, \ldots, 9\}$. Note that

$$x = \sum_{i=k}^{n-1} a_i \cdot 10^{-i} + a_n \cdot 10^{-n} + x_{n+1} = \sum_{i=k}^{n} a_i \cdot 10^{-i} + x_{n+1}$$

so the inductive hypothesis is true for n. Thus we can find all a_i with $i \geq k$ by induction.

All that is left is to show that the sum $\sum_{i=k}^{\infty} a_i \cdot 10^{-i}$ converges to x. Choose $\epsilon > 0$, and let s_n be the partial sum $\sum_{i=k}^{n} a_i \cdot 10^{-i}$. By the inductive proof above, for each $n \geq k$ we have

$$x - s_n = x_{n+1} < 10^{-n}$$

so we merely have to choose an n such that $10^{-n} < \epsilon$; we can find such an n in the same way as we found k at the beginning of this proof. Thus

$$x = \lim_{n\to\infty} s_n = \sum_{i=k}^{\infty} a_i \cdot 10^{-i}.$$

❐

Now we show that this decimal expansion is usually unique.

CLAIM. *The a_i from the previous claim are uniquely determined unless there exists an $n \in \mathbf{N}$ with $10^n \cdot x \in \mathbf{N}$, in which case there are exactly two different decimal expansions.*

PROOF.

First we require a simple lemma.

LEMMA. *If $a_i \in \{0, 1, \ldots, 9\}$ for all $i \geq k$ for some k then*

$$\sum_{i=k}^{\infty} a_i \cdot 10^{-i} \leq 10^{-k+1}$$

with equality if and only if each $a_i = 9$.

PROOF.

Let x be the above sum. First note that the sum converges for any choice of a_i, since it is bounded above by the sum

$$\sum_{i=k}^{\infty} 10^{-i+1} = 10^{-k+1} \sum_{i=0}^{\infty} 10^{-i} = 10^{-k+1} \frac{1}{1 - \frac{1}{10}} = \frac{10^{-k+2}}{9}$$

which converges. The largest possible value of x occurs exactly when each $a_i = 9$; indeed, if $a_n < 9$ then we could add $a_n \cdot 10^{-n}$ to x to obtain a larger number expressed in the same form. So the entire lemma reduces to showing that $x = 10^{-k+1}$ when each $a_i = 9$. This is not so hard:

$$\sum_{i=k}^{\infty} 9 \cdot 10^{-i} = 9 \cdot \sum_{i=k}^{\infty} \cdot 10^{-i} = 9 \cdot \frac{10^{-k+1}}{9} = 10^{-k+1}$$

by the above calculation.

❐

Suppose that x has two decimal expansions, i.e. we can find appropriate a_i, k and a'_i, k' as above such that

$$x = \sum_{i=k}^{\infty} a_i \cdot 10^{-i} = \sum_{i=k'}^{\infty} a'_i \cdot 10^{-i}.$$

For simplicity, assume that $k = k'$; we can do this by taking both to be $\min\{k, k'\}$, and setting a_i or a'_i to be zero where they're not defined. Let n be the minimal index such that $a_n \neq a'_n$ (note that such an n is guaranteed by the well-ordering principle) and assume $a'_n < a_n$. We have

$$0 = \sum_{i=k}^{\infty} a_i \cdot 10^{-i} - \sum_{i=k}^{\infty} a'_i \cdot 10^{-i} = \sum_{i=n}^{\infty} (a_i - a'_i) \cdot 10^{-i}$$

so that

$$(a_n - a_n') \cdot 10^{-n} = \sum_{i=n+1}^{\infty} (a_i' - a_i) \cdot 10^{-i} \iff a_n - a_n' = 10^n \cdot \sum_{i=n+1}^{\infty} (a_i' - a_i) \cdot 10^{-i}.$$

But by the lemma above, $\sum_{i=n+1}^{\infty} (a_i' - a_i) \cdot 10^{-i} \leq 10^{-n}$ with equality if and only if each $a_i' - a_i = 9$, so

$$10^n \cdot \sum_{i=n+1}^{\infty} (a_i' - a_i) \leq 1.$$

But $a_n - a_n' \geq 1$ so $a_n = a_n' + 1$ and each $a_i' - a_i = 9$, which only happens when each $a_i' = 9$ and each $a_i = 0$ (for $i > n$). Thus $10^n \cdot x \in \mathbf{N}$ since

$$10^n \cdot x = 10^n \cdot \sum_{i=k}^{\infty} a_i \cdot 10^{-i} = \sum_{i=k}^{n} a_i \cdot 10^{n-i}$$

which is a positive integer. Thus, given a decimal expansion of x (as guaranteed by the first part of the problem), we have proved that there is at most one other decimal expansion, and have classified all cases when that can happen.

Conversely, if $10^n \cdot x \in \mathbf{N}$ for some $n \in \mathbf{N}$ then x has a natural expression as a terminating decimal, and also as a decimal trailing an infinite number of nines; by the above, those are the only two expressions of x, so in this case, x has exactly two decimal expansions.

❒

Notes on this problem:

(1) In almost every notational convention (with the only exception I know of being doing differential geometry using Einstein summation notation), you *can not* define a sequence $\{a_i\}$ and then define $\{a_j\}$ to be a different sequence. The sequence $\{a_j\}$ is *the same* sequence, except using the letter j instead of i to denote the indices. Trying to redefine it makes what you subsequently write nearly impossible to understand. Consider what would happen if you wanted to add the ith element of the first sequence to the kth — you would write $a_i + a_k$, but now your reader is confused as to why a_i and a_j are different sequences but a_k is not. Always remember when writing mathematics that someone will read it at some point.

(2) See "A Note on Proofs" on the website.

(3) See "A Note on Proofs" on the website. (In case you're wondering why I included these two notes at all, it's because I refer people to note numbers on graded problem sets.)

(4) *Always define your variables* in the same sentence that you first use them. The quickest way to confuse a reader is to introduce a symbol whose object is unclear. You're only allowed to do that in literature.

(5) A more specific note: just because *your* algorithm gave a unique decimal expansion, it doesn't mean that that expansion is unique — some other algorithm may have come up with a different expansion. Therefore, uniqueness really does have to be proved separately.

(6) You weren't allowed to assume the existence of decimal expansions in order to do this problem, since this problem is the proof that decimal expansions exist.

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