MATH 23b, SPRING 2004
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
(Final Version) Homework Assignment # 2
Due: February 20, 2004

Please turn in four separate sets labelled A through D.

1. Read Sections 1.7–1.8 from Edwards.
2. (A) For a set $A \subset \mathbb{R}^n$, show that $\partial A = \overline{A} \setminus A^\circ$.
3. (*) For a set $A \subset \mathbb{R}^n$, show that $\partial A = \overline{A} \cap \overline{A^c}$.
4. (B) Let $V = \mathbb{R}^2$, and consider the following subsets:

$$A = \mathbb{Q} \times \mathbb{Q} = \{(x, y) \in \mathbb{R}^2 \mid x, y \in \mathbb{Q}\}$$

$$B = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$$

   For the following, recall that $\overline{S}$, $S^\circ$, and $S^c$ denote the closure, interior (see HW #1.7), and complement, respectively, of $S$.

   (a) Find $A^\circ$, $(A^c)^\circ$, $\overline{A}$, $\overline{A^c}$, and $\overline{A} \cap \overline{A^c}$.
   (b) Find $B^\circ$, $(B^c)^\circ$, $\overline{B}$, $\overline{B^c}$, and $\overline{B} \cap \overline{B^c}$.
   (c) Find $A \cap B$, $\overline{A \cap B}$ and $(A \cap B)^\circ$. In light of problem # 2 above, what does this say about the boundary of $A \cap B$?

5. (*) Let $S = \{(x, \sin(\frac{1}{x})) \mid x > 0\} \subset \mathbb{R}^2$. Find $\overline{S}$.
6. (C) A subset $S$ in a metric space or a normed vector space is called **discrete** if, for every $x \in S$, there is some $\varepsilon > 0$ such that $B_\varepsilon(x) \cap S = \{x\}$, that is, the only intersection between the ball and the set is the point itself.

   (a) Show that every $f : S \longrightarrow \mathbb{R}$ is continuous if $S$ is discrete.
   (b) Consider $\mathbb{R}^n$ with its usual inner product. Show that every closed, bounded, and discrete set is finite, and give examples why each of these conditions is necessary.
   (c) Show that $\mathbb{Z} \subset \mathbb{R}$ is discrete.

7. (D) Let $A \subset \mathbb{R}^n$ be compact, and suppose that $B \subset A$ is closed. Use the "open cover" definition to show that $B$ is compact.
8. (D) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuous, and let $A \subset \mathbb{R}^n$ be compact. Use the "open cover" definition to show that $f(A)$ is compact.

9. (deferred) The Cantor Intersection Theorem states that if $\{Q_n\}_{n=1}^{\infty}$ is a nested (so that $Q_{n+1} \subset Q_n, \forall n$) collection of non-empty, bounded, closed sets in $\mathbb{R}^n$, then $S = \bigcap_{n=1}^{\infty} Q_n$ is also non-empty, bounded, and closed. Illustrate that the hypotheses are necessary by giving examples of the following cases:

   (a) the $Q_n$ are nested, non-empty, and bounded, but not closed, and $\bigcap_{n=1}^{\infty} Q_n = \emptyset$.

   (b) the $Q_n$ are nested, non-empty, and closed, but not bounded, and $\bigcap_{n=1}^{\infty} Q_n = \emptyset$.