MATH 23a, FALL 2003
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
(Final Version) Homework Assignment #3
Due: October 10, 2003

1. Read Sections 3.3–3.5 and begin Chapter 5 of Schneider and Barker, and read Sections 1.4–1.5 of Edwards.

2. (A) Let V be a vector space, and let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset V$ be a collection of linearly independent vectors. Show that no collection of $n-1$ vectors spans V.

3. (B) Let S be some set with finite cardinality n. (For simplicity, you may assume that $S = \{1, 2, \ldots, n\}$.) Find a basis for the vector space of functions $V = \{f : S \to \mathbb{R}\}$, where addition and scalar multiplication are defined as for functions of one real variable.

4. (C) Let $V = \{(a_0, a_1, a_2, \ldots) \mid a_i \in \mathbb{R}\}$ be the vector space of all infinite sequences of real numbers. Let W be the subspace of V consisting of all *arithmetic* sequences. Find a basis for W, and determine the dimension of W. (*A sequence is arithmetic is there is some constant c such that $a_{n+1} = a_n + c$ for all $n \geq 0$.*)

5. (D) Let p be a fixed prime number, and let $V = (\mathbb{Z}/p\mathbb{Z})^n$ with $n \geq 2$.
   (a) How many distinct one-dimensional subspaces does V have?
   (b) How many distinct two-dimensional subspaces does V have?

6. (*) Let $F = \mathbb{Z}/7\mathbb{Z}$. Let $\mathbf{u} = (1, 0, 6)$ $\mathbf{v} = (1, 2, 1)$, and $\mathbf{w} = (2, 1, 3)$ be three vectors in F^3, that is, the set of ordered triples with coordinates in F. Find coefficients $a, b, c \in F$ to express the vector $\mathbf{x} = (1, 2, 3)$ as a linear combination $\mathbf{x} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$.

7. (deferred) Let $P_n(\mathbb{R}) = \{p(x) = a_0 + a_1 x + \cdots + a_n x^n \mid a_i \in \mathbb{R}, \forall i\}$ be the vector space of all polynomials of degree less than or equal to n. Consider the map $L : P_n(\mathbb{R}) \to \mathbb{R}$ defined by $L(p) = \int_0^1 p(x)\, dx$.
   (a) Show that L is a linear map.
   (b) Determine $Im(L)$, and find a basis.

8. (E) Let U and W be subspaces of a vector space V. We define two new subspaces as follows:

$$U + W = \{\mathbf{u} + \mathbf{w} \mid \mathbf{u} \in U, \mathbf{w} \in W\}$$
$$U \cap W = \{\mathbf{v} \in V \mid \mathbf{v} \in U \text{ and } \mathbf{v} \in W\}$$

(a) (*) Convince yourself that both $U + W$ and $U \cap W$ are, in fact, subspaces.

(b) Show that if $dim(V) < \infty$, then

$$dim(U + W) = dim(U) + dim(W) - dim(U \cap W).$$

9. (*) Show that if W is a subspace of V and $dim(V) < \infty$,

   then $dim(W) \leq dim(V)$. (Part of this problem is showing that W *has* a basis. Do this constructively by choosing vectors successively.)

10. (deferred) In this problem, we consider the *shift* operator. Consider the linear map $S : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ which acts as follows:

$$S(x, y, z) = (0, x, y).$$

   Find the kernel and image of S, and verify that

$$dim(Ker(S)) + dim(Im(S)) = dim(\mathbb{R}^3).$$

11. (deferred) We generalize the notion of the *shift* operator. Let V be the vector space of all infinite sequences of real numbers as in Problem #2.10 (and #3.4), and consider the linear maps $S : V \rightarrow V$ and $T : V \rightarrow V$, where S and T act as follows:

$$S(a_0, a_1, a_2, \ldots) = (0, a_0, a_1, a_2, \ldots)$$

$$T(a_0, a_1, a_2, \ldots) = (a_1, a_2, \ldots)$$

   (a) Find the kernel and image of S. How does the result about the dimensions of kernels and images apply?

   (b) Show that $T \circ S = I$ but that $S \circ T \neq I$, where $I : V \longrightarrow V$ is the identity map.

   (c) Which of S and T is onto? Which is one-to-one? Which is invertible? Explain.