

MATH 23b, SPRING 2004
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Homework Assignment # 3
Due: February 27, 2004

Homework Assignment #3 (Final Version)

1. Read Edwards, Sections 1.8 and 2.1. Supplementary: Fitzpatrick, Chapters 10 and 11.
2. (A) The Cantor Intersection Theorem states that if $\{Q_n\}_{n=1}^{\infty}$ is a nested (so that $Q_{n+1} \subset Q_n, \forall n \in \mathbb{N}$) collection of non-empty, bounded, closed sets in \mathbb{R}^n , then $S = \bigcap_{n=1}^{\infty} Q_n$ is also non-empty, bounded, and closed. Illustrate that the hypotheses are necessary by giving examples of the following cases:
 - (a) the Q_n are nested, non-empty, and bounded, but not closed, and $\bigcap_{n=1}^{\infty} Q_n = \emptyset$.
 - (b) the Q_n are nested, non-empty, and closed, but not bounded, and $\bigcap_{n=1}^{\infty} Q_n = \emptyset$.
3. (*) In the proof of the Lindelof Covering Theorem, we needed the following fact:

Fact: If $A \subset \mathbb{R}^n$ is an open set and $\mathbf{x} \in A$, then there is some ball $B_\varepsilon(\mathbf{y}) \subset A$ with rational center (not necessarily \mathbf{x} !) and rational radius ε such that $\mathbf{x} \in B_\varepsilon(\mathbf{y})$.

For the remainder of the homework set, consider $V = \mathbb{R}^n$. Recall that we define the collection of linear transformations (and their matrices with respect to the standard basis) from V to V to be:

$$M_n(\mathbb{R}) = \{A : V \longrightarrow V \mid A \text{ is linear}\},$$

and that we may assume that $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ has the structure of Euclidean space. We also define the collection of invertible linear transformations, called the *general linear group*, to be:

$$GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid A \text{ is invertible}\}.$$

We further define two special subgroups of the general linear group, called the *special linear group* and the *orthogonal group*, respectively, as follows:

$$SL_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) \mid \det(A) = 1\}$$

$$O_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) \mid A^t A = I\}$$

(Recall that A^t is the transpose of the matrix A .)

4. (*) Show that $\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$ is continuous.
(Hint #1: Use the theorems, not the definition.
Hint #2: This part is important for the rest of the questions.)
5. (A) Show that $GL_n(\mathbb{R})$ is open as a subset of $M_n(\mathbb{R})$.
(Hint: Use #4.)
6. (B) Show that $SL_n(\mathbb{R})$ is closed but not compact.
7. (B) (Sorry about the order switch!)
Show that $O_n(\mathbb{R})$ is compact in $M_n(\mathbb{R})$ by showing that:
 - (a) $O_n(\mathbb{R})$ is closed in $M_n(\mathbb{R})$.
 - (b) $O_n(\mathbb{R})$ is bounded in $M_n(\mathbb{R})$. (Hint: What do you know about the columns of an orthogonal matrix?)
8. (C) Show that the closure of $GL_n(\mathbb{R}) \subset M_n(\mathbb{R})$ is all of $M_n(\mathbb{R})$.
9. (D) Let X be the collection of diagonalizable matrices in $M_2(\mathbb{R})$.
Is X open, closed, or neither? Explain.