MATH 23b, SPRING 2004
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Homework Assignment # 4
Due: March 5, 2004

Homework Assignment #4 (Final Version)

1. Read Edwards, Section 2.1–2.3.

2. (A)

   (a) Let $f : \mathbb{R} \longrightarrow \mathbb{R}^n$ and $g : \mathbb{R} \longrightarrow \mathbb{R}^n$ be two differentiable curves, with $f'(t) \neq 0$ and $g'(t) \neq 0$ for all $t \in \mathbb{R}$. Suppose that $\mathbf{p} = f(s_0)$ and $\mathbf{q} = g(t_0)$ are closer than any other pair of points on the two curves. Prove that the vector $\mathbf{p} - \mathbf{q}$ is orthogonal to both velocity vectors $f'(s_0)$ and $g'(t_0)$.

   (Hint: The point (s_0, t_0) must be a critical point for the function $\rho : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $\rho(s,t) = |f(s) - g(t)|^2$.)

   (b) Apply the result of part (a) to find the closest pair of points of the "skew" straight lines in $\mathbb{R}^3$ defined by $f(s) = (s, 2s, -s)$ and $g(t) = (t+1, t-2, 2t+3)$.

3. (A) Consider a particle which moves on a circular helix in $\mathbb{R}^3$ with position vector given by (all scalars non-zero):

$$\varphi(t) = (a\cos\omega t, a\sin\omega t, b\omega t).$$

   (a) (*) Graph $Im(\varphi)$ using Mathematica!

   (b) Show that the speed of the particle is constant.

   (c) Show that the velocity vector makes a constant nonzero angle with the z-axis.

   (d) If $t_1 = 0$ and $t_2 = \frac{2\pi}{\omega}$, notice that $\varphi(t_1) = (a, 0, 0)$ and $\varphi(t_2) = (a, 0, 2\pi b)$, so the vector $\varphi(t_2) - \varphi(t_1)$ is vertical. Conclude that the equation

$$\varphi(t_2) - \varphi(t_1) = (t_2 - t_1)\varphi'(\tau)$$

   cannot hold for any $\tau \in (t_1, t_2)$. Thus the Mean Value Theorem does not hold for vector-valued functions.

   (e) (*) Review: Look up the Mean Value Theorem!

4. (B) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by $f(x,y) = \frac{x^2y}{x^2+y^2}$ unless $x = y = 0$, in which case $f(0,0) = 0$.

   (a) (*) Graph f using Mathematica! *(The picture of this function was distributed in class, but it will be useful for you to be able to generate it yourself.)*

   (b) Show that $D_{\mathbf{v}}f(0,0)$ exists for all $\mathbf{v} \in \mathbb{R}^2$ by direct computation. *(Hint: You should conclude that $D_{\mathbf{v}}f(0,0) = f(\mathbf{v})$.)*

   (c) Show that f satisfies the homogeneous relation $f(t\mathbf{v}) = tf(\mathbf{v})$ for all $t \in \mathbb{R}$ and all $\mathbf{v} \in \mathbb{R}^2$.

   (d) Show that any *differentiable* function $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfying the homogeneous relation $g(t\mathbf{v}) = tg(\mathbf{v}), \forall t \in \mathbb{R}, \forall \mathbf{v} \in \mathbb{R}^n$ and $g(\mathbf{0}) = 0$ also satisfies the relation $g(\mathbf{v}) = \nabla g(\mathbf{0}) \cdot \mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^n$ and hence must be *linear.*

   (e) Conclude that f possesses directional derivatives in all directions at $(0,0)$, but that f is *not* differentiable at $(0,0)$.

5. (C) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by $f(x,y) = x^2 \sin(1/x) + y^2$ for $x \neq 0$ and $f(0,y) = y^2$.

   (a) Show that f is continuous at $(0,0)$.

   (b) Find the partial derivatives of f at $(0,0)$.

   (c) Show that f is differentiable at $(0,0)$.

   (d) Show that $D_1 f$ is *not* continuous at $(0,0)$.

6. (D) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by $f(x,y) = (\sin(x-y), \cos(x+y))$. Find the equations of the tangent plane in $\mathbb{R}^4$ to the graph of f at the point $(\frac{\pi}{4}, \frac{\pi}{4}, 0, 0)$.

7. (D) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$. Prove that if f is differentiable at $\mathbf{a}$, then f is continuous at $\mathbf{a}$.