Solutions to HW 4 Part B

This set was pretty hard; the median score was merely an 8.5 and the mean was 8.3 out of 10. Anyway, here are the solutions:

3a. Observe that f is harmonic iff $\frac{\partial^2 f}{\partial^2 x} + \frac{\partial^2 f}{\partial^2 y} = 0$. However, $\frac{\partial f}{\partial x} = 2ax + by + d$, so $\frac{\partial^2 f}{\partial^2 x} = 2a$ and $\frac{\partial f}{\partial y} = by + 2cx + e$, so $\frac{\partial^2 f}{\partial^2 y} = 2c$ and f is harmonic iff $2a + 2c = 0$, which holds iff $a = -c$.

NOTE: Most people got the full two points for this question. However, I did take off a point if you didn't show any work in your computations.

b. By the definition of the norm, it follows that

$$f(x) = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1-n/2}.$$

Therefore, if $n \neq 2$ and j is a positive integer with $1 \leq j \leq n$, then

$$\frac{\partial f}{\partial j} = (2-n)x_j(x_1^2 + x_2^2 + \ldots + x_n^2)^{-n/2}$$

so

$$\frac{\partial^2 f}{\partial^2 j} = (2-n)(x_1^2+x_2^2+\ldots+x_n^2)^{-n/2} - (2-n)(n)x_j^2(x_1^2+x_2^2+\ldots+x_n^2)^{-1-n/2}.$$

Therefore,

$$\begin{aligned} \nabla^2 f &= n(2-n)(x_1^2 + x_2^2 + \ldots + x_n^2)^{-n/2} - n(2-n)(x_1^2 + x_2^2 + \ldots + x_n^2)(x_1^2 + x_2^2 + \ldots + x_n^2)^{-1-n/2} \\ &= n(2-n)(x_1^2 + x_2^2 + \ldots + x_n^2)^{-n/2} - n(2-n)(x_1^2 + x_2^2 + \ldots + x_n^2)^{-n/2} \\ &= 0 \end{aligned}$$

for all nonzero vectors x. However, if $n = 2$ then $f(x) = 1$ for all nonzero vectors x; because it is constant at all such values it is clearly harmonic in this case as well.

NOTE: Almost no one took into account the case where $n = 2$ (this case needed to be dealt with separately because evaluating the first partials of f required the Power Rule; when $n = 2$ the exponent was 0 which meant that the Power Rule did not apply); doing so cost you one point out of four.

c. Observe that $\frac{\partial f}{\partial x} = D_1 g(e^x \cos y, e^x \sin y)e^x \cos y + D_2 g(e^x \cos y, e^x \sin y)e^x \sin y$, so

$$\begin{aligned} \frac{\partial^2 f}{\partial^2 x} &= D_1(D_1 g)(e^x \cos y, e^x \sin y)(e^{2x}(\cos y)^2) + D_2(D_1 g)(e^x \cos y, e^x \sin y)(e^{2x}(\cos y \sin y)) \\ &\quad + D_1 g(e^x \cos y, e^x \sin y)e^x \cos y + D_1(D_2 g(e^x \cos y, e^x \sin y))(e^{2x}(\sin y \cos y)) \\ &\quad + D_2(D_2 g(e^x \cos y, e^x \sin y))(e^{2x}(\sin y)^2) + D_2 g(e^x \cos y, e^x \sin y)(e^x \sin y). \end{aligned}$$

Similarly, $\frac{\partial f}{\partial y} = -D_1 g(e^x \cos y, e^x \sin y)e^x \sin y + D_2 g(e^x \cos y, e^x \sin y)e^x \cos y$ so

$$\begin{aligned}\frac{\partial^2 f}{\partial^2 y} &= D_1(D_1 g)(e^x\cos y, e^x \sin y)(e^{2x}(\sin y)^2) - D_2(D_1 g)(e^x \cos y, e^x \sin y)(e^{2x}(\sin y \cos y)) \\ &\quad -D_1 g(e^x \cos y, e^x \sin y)e^x \cos y - D_1(D_2 g(e^x \cos y, e^x \sin y))(e^{2x}(\cos y \sin y)) \\ &\quad +D_2(D_2 g(e^x \cos y, e^x \sin y))(e^{2x}(\cos y)^2) - D_2 g(e^x \cos y, e^x \sin y)(e^x \sin y).\end{aligned}$$

It follows that

$$\begin{aligned}\nabla^2 f &= e^{2x}(D_1(D_1 g)((\cos y)^2 + (\sin y)^2) + D_1(D_2 g)(\sin y \cos y - \sin y \cos y) \\ &\quad +D_2(D_1 g)(\sin y \cos y - \sin y \cos y) + D_2(D_2 g)((\sin y)^2 + (\cos y)^2) \\ &\quad +e^x(D_1 g(e^x \cos y, e^x \sin y)(e^x \cos y - e^x \cos y + e^x(D_2 g(e^x \cos y, e^x \sin y))(e^x \sin y - e^x \cos y) \\ &= e^{2x}(D_1(D_1 g) + D_2(D_2 g));\end{aligned}$$

because $D_1(D_1 g)+D_2(D_2 g)) = 0$ by assumption (as g is harmonic), $\nabla^2 f = e^{2x} \cdot 0 = 0$ so f is harmonic as well.

NOTE: Many people wrote expressions like:

$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x}$

$g(e^x cosy, e^x siny)e^x cosy + \frac{\partial f}{\partial y}$

$g(e^x cosy, e^x siny)e^x siny.$

Expressions like these were wrong because expressions such as $D_1(e^x \cos y, e^x \sin y)$ (which takes the partial derivative with respect to the first component of the domain of g) are different from expressions such as $\frac{\partial f}{\partial x}(e^x \cos y, e^x \sin y)$ (which takes the partial derivative with respect to the first component of the domain of f). I only took off a half a point out of four for this mistake because it could easily be just a notational mistake. However, if people compounded on that mistake in differentiating the second derivatives by differentiating the wrong partials, I took off further points.

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