

MATH 23b, SPRING 2004
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Homework Assignment # 5
Due: March 19, 2004

Homework Assignment #5 (Final Version)

1. Read Edwards, Sections 2.3–2.5.
Next week, we will also be discussing some of Chapter 3.

2. (*) More on gradients:

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$, and $g : \mathbb{R}^n \rightarrow \mathbb{R}$ be differentiable. Show that:

- (a) $\nabla(f + g) = \nabla f + \nabla g$.
- (b) $\nabla(fg) = (\nabla f)g + f(\nabla g)$.
- (c) $\nabla(f^m) = mf^{m-1}\nabla f$, for any positive integer m .
- (d) Determine a formula for $\nabla(\frac{f}{g})$ when $g(\mathbf{x}) \neq 0$.

3. (A) Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function with continuous second-order partial derivatives (so that, in particular, our theorem about cross-partials applies, and $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$).

With $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$, the usual gradient of f , we make the following definitions:

- $\|\nabla f\|^2 = (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2$ is the norm (squared) of the gradient of f .
- $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$ is the *Laplacian* of f .

Finally, let $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by $g(r, \theta) = f(r \cos \theta, r \sin \theta)$.

- (a) Show that $\|\nabla f\|^2 = (\frac{\partial g}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial g}{\partial \theta})^2$.
- (b) Show that $\nabla^2 f = \frac{\partial^2 g}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 g}{\partial \theta^2} + \frac{1}{r} \frac{\partial g}{\partial r}$.

4. (B) Following up on problem #3, if $f : \mathbb{R}^n \rightarrow \mathbb{R}$ has continuous second-order partials, the *Laplacian* of f is defined to be $\nabla^2 f = \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2}$.

With f as above, we say that f is *harmonic* on the open set $U \subset \mathbb{R}^n$ provided that $\nabla^2 f(\mathbf{x}) = 0$, $\forall \mathbf{x} \in U$.

- (a) Find a (simple) condition on the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $f(x, y) = ax^2 + bxy + cy^2 + dx + ey + k$ that makes f harmonic.
- (b) Show that $f : \mathbb{R}^n \rightarrow \mathbb{R}$ defined by $f(\mathbf{x}) = \frac{1}{\|\mathbf{x}\|^{n-2}}$ is harmonic on $U = \mathbb{R}^n \setminus \{\mathbf{0}\}$.

(c) Show that if $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ is harmonic, then $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $f(x, y) = g(e^x \cos y, e^x \sin y)$ is also harmonic.

5. (C) Define $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ by $f(x, y) = x \cdot y \cdot \frac{x^2 - y^2}{x^2 + y^2}$ for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$.

(a) (*) Graph $z = f(x, y)$ using Mathematica.

(b) Show that $D_1 f(0, y) = -y$ and $D_2 f(x, 0) = x$ for all $x, y \in \mathbb{R}$.

(c) Show that $D_2 D_1 f(0, 0)$ and $D_1 D_2 f(0, 0)$ exist but are not equal.

6. (C) Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is differentiable and has an inverse function $f^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ which is also differentiable. Show that:

$$[J(f^{-1})](\mathbf{a}) = [(Jf)(f^{-1}(\mathbf{a}))]^{-1}.$$

Sorry for all the parentheses, but I am trying to make this clear. On the left-hand side, we are taking the Jacobian of f^{-1} and evaluating at \mathbf{a} . On the right-hand side, we are taking the Jacobian of f and evaluating at $f^{-1}(\mathbf{a})$, and then taking the inverse (as a matrix) of that.

7. (D) Recall that we have an isomorphism of vector spaces $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$.

(a) Consider the determinant map $\det : M_n(\mathbb{R}) \rightarrow \mathbb{R}$, and find $\nabla(\det)(A)$, expressed in terms of $A = [a_{ij}]$.

(b) Consider the function $f : M_n(\mathbb{R}) \rightarrow M_n(\mathbb{R})$ given by $f(A) = A^2$. Show that $Jf_A(H) = AH + HA$.

8. (*) Consider the function $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ given by $f(\mathbf{x}) = \|\mathbf{x}\|\mathbf{x}$. Determine whether or not f is differentiable at $\mathbf{0}$. If not, why not? If so, find the first-order partial derivatives of f at $\mathbf{0}$. (Do the second-order partial derivatives of f exist at $\mathbf{0}$?)

9. (*) It may have occurred to you to ask whether a function could be differentiable at a point but not on any open set containing that point. Consider the function $f_3 : \mathbb{R} \rightarrow \mathbb{R}$ given by:

$$f_3(x) = \begin{cases} 0 & , \text{ if } x \notin \mathbb{Q} \\ \frac{1}{q^3} & , \text{ if } x = \frac{p}{q} \text{ in lowest terms} \end{cases}$$

(a) Show that f_3 is not continuous at any $x \in \mathbb{Q}$.

(b) Show that f_3 is differentiable at $x = \sqrt{2}$.