MATH 23B, SOLUTION SET FOR PS 5, PART B

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If you don't understand anything about any of the solutions here, or if you spot mistakes, feel free to e-mail me.

Problem 3

(a) Given $\phi : \mathbb{R}^3 \to \mathbb{R}$ and $\psi : \mathbb{R}^3 \to \mathbb{R}$, we define $F : \mathbb{R}^3 \to \mathbb{R}$ by

$$F(x) = (\phi(x), \psi(x), 1 + \phi(x)\psi(x) + \phi(x)^3), \qquad x \in \mathbb{R}^3.$$

The Jacobian matrix looks like

$$JF = \begin{pmatrix} \nabla\phi(x) \\ \nabla\psi(x) \\ \nabla F_3(x) \end{pmatrix}.$$

Since,

$$\partial F_3/\partial x_1 = \frac{\partial \phi}{\partial x_1}\psi + \frac{\partial \psi}{\partial x_1}\phi + 3\phi(x)^2\frac{\partial \phi}{\partial x_1},$$

we can write (abusing notation and symmetry),

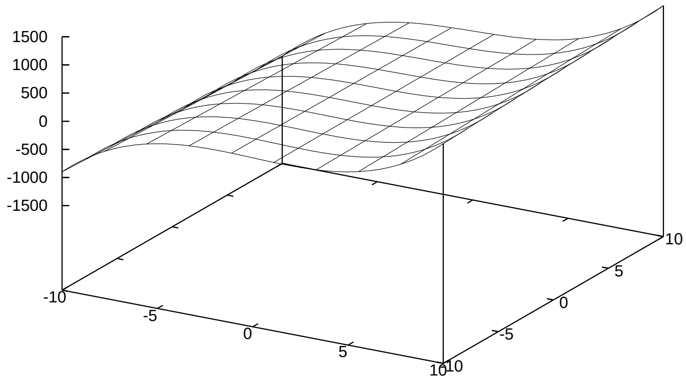
$$\nabla F_3(x) = \psi(x)(\nabla\phi(x)) + \phi(x)(\nabla\psi(x)) + 3\phi(x)^2(\nabla\phi(x)).$$

At any point $x \in \mathbb{R}^3$, $\phi(x)$, $\psi(x)$, and $3\phi(x)^2$ are constant, so the third row is linearly dependent on on the first two rows. This implies that $\det JF(x) = 0$. Hence, the conditions of the Inverse Function Theorem are never satisfied by F.

(b) As far as geometric intuition is conerned, consider $h : \mathbb{R}^3 \to \mathbb{R}^2$ mapping $x \in \mathbb{R}^3 \mapsto (\phi(x), \psi(x))$ and $k : \mathbb{R}^2 \to \mathbb{R}^3$ mapping $(u, v) \mapsto (u, v, 1 + uv + u^3)$. It is clear that $F = k \circ h$. So, if we trace through, we have $\mathbb{R}^3 \to \mathbb{R}^2 \to \mathbb{R}^3$.

One way to see that there is no neighborhood U in $\mathbb{R}^3$ such that F^{-1} is well-defined on U is to consider modifying $F(x)$ in the third coordinate. That is, let $F(x) = (x_1, x_2, x_3)$. Then consider $y = (x_1, x_2, x_3 + \epsilon)$. Given any U open, there exist ϵ such that $y \in U$. But y is not in the image of F for $\epsilon \neq 0$ because there is only one compatible third coordinate for any pair of x_1 and x_2, determined by k.

Lukasz showed a very nice picture giving some more intuition. The image of k is the surface $\{(x, y, z) \mid z = 1 + xy + x^3\}$. This surface contains no open balls in $\mathbb{R}^3$, because we can't jump off of the plane.



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