HW 6 Part D Solutions

This set was a lot harder than usual; however, the class did really well this week (among the 52 students who submitted solutions to this problem) as the average score was a 9.3/10 and the median was 9.5/10. Also, the upper right-hand corner was removed from one paper (and the name did not appear elsewhere); if this is yours email me at meyerson@fas.harvard.edu and I will give you the points you earned.

5. We shall begin by calculating the first and second order partial derivatives of f at $a = (1, 0, -1)$: $\frac{\partial f}{\partial x} = y^2z^3$, which equals 0 at a (as $y = 0$), $\frac{\partial f}{\partial y} = 2xyz^3$, which also equals 0 at a, and $\frac{\partial f}{\partial z} = 3xy^2z^2$, which also equals 0 at a.

   Now, in calculating the second order partial, we have $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(y^2z^3) = 0$, $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(y^2z^3) = 2yz^3$, which equals 0 at a, $\frac{\partial^2 f}{\partial z \partial x} = \frac{\partial}{\partial z}(y^2z^3) = 3y^2z^2$, which equals 0 at a, $\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(2xyz^3) = 2xz^3$, which equals -2 at a, $\frac{\partial^2 f}{\partial z \partial y} = \frac{\partial}{\partial z}(2xyz^3) = 6xyz^2$, which equals 0 at a, and $\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}(3xy^2z^2) = 6xy^2z$, which also equals 0 at a.

   In addition, because $f \in C^2$ (in fact, $f \in C^\infty$), the higher-order partial derivatives are independent of the order in which the derivatives are taken; therefore, $\frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial^2 f}{\partial x \partial z}$, and $\frac{\partial^2 f}{\partial y \partial z}$ are all equal to 0 at a. Observe that $f(a) = 0$ as well; therefore, the second-order Taylor polynomial T_2 for f at a is equal to $f(a + h) = f(a) + \nabla f(a) \cdot h + q(h)$; the RHS simplifies to $q(h)$ as $f(a)$ and $\nabla f(a)$ are all 0. However, as the only nonzero partial derivative of f at a is $\frac{\partial^2 f}{\partial y^2}$, it follows that if h is coordinatized as (h_1, h_2, h_3) then $q(h) = \frac{1}{2} \cdot \frac{\partial^2 f}{\partial^2 y}(h_2)^2$ which equals $\frac{1}{2}(-2(h_2)^2) = -h_2^2$. This implies that $T_2(a + h) = -h_2^2$.

   This implies that

$$\begin{aligned}
R_2(h) &= f(a+h) - T_2(a+h) \\
&= (1+h_1)(0+h_2)^2(-1+h_3)^3 - h_2^2 \\
&= h_2^2(1+h_1)(-1+h_3)^3 + h_2^2 \\
&= h_2^2 - h_2^2 - h_2^2 h_1 + h_2^2 h_3 p(h),
\end{aligned}$$

   where $p(h)$ is a polynomial in h_1, h_2, and h_3; therefore, $R_2(h) = -h_2^2h_1 + h_2^2h_3p(h)$ which implies that $|R_2(h)| \le ||h||^3(1 + p(h))$.

   Therefore, $|R_2(h)/||h||^2| \le ||h||(1 + p(h))$,

   which clearly approaches 0 as h approaches 0; this implies that

   $\lim_{||h|| \to 0}((R_2(h))/||h||^2) = 0$ as well.

NOTE: Thinking of T_2 as a function of (x, y, z) uses the exact same method (although the math is a little easier as $f(x, y, z) = xy^2z^3$ while $T_2(x, y, z) = y^2$).

I only required you to show that one of the two limits mentioned in the problem equalled zero to receive full credit; if someone showed both methods I graded each method separately and then gave him or her them the highest of the two grades as my score for this problem.

The biggest problem people had here was in evaluating limits; a lot of people said something like "all the terms in the numerator are of degree three or greater in h_1, h_2, h_3 and the denominator is of degree two in h_1, h_2, h_3; therefore, the limit is zero as h goes to zero".

A counterexample to this statement is the limit of $f(h_1, h_2, h_3) = h_1^3/h_3^2$ as h goes to zero; consider the behavior of the function on the path $(t, 0, t^2)$ or some other path sufficiently close to the h_1-axis to see why the limit cannot possibly approach zero.

A few people also said something like "the numerator is the product of three things which approach zero but the denominator is the product of two things which approach zero; therefore, the limit also approaches zero". This is also wrong; as a counterexample check the limit of $(t * t^2 * t^3)/(t * t^{10})$ as t approaches 0; although the numerator and denominator satisfy the appropriate conditions, the fraction equals t^{-5} which clearly does not approach zero as t approaches zero.

The cost for making either of those two mistakes (or a similar limit-related mistake) was two points out of ten.

Finally, some people stated slightly stronger inequalities about the limits than were actually true; these invariably involved using a $>$ sign rather than a $\geq$. For example, as h approaches 0 it is not true that all of h_1^2, h_2^2, h_3^2 are greater than zero along the path of approach; it is merely required that all three be greater than or equal to zero. To see this try traveling along the h_2-axis; here, $h_1^2 = h_3^2 = 0^2 = 0$ at each point along this axis. Such an innapropriate strengthening only cost half a point though.