Solution Set 6E

7. Since f is $\mathcal{C}^3$, Taylor's Theorem tells us that near $\mathbf{a}$, we have

$$f(\mathbf{a} + \mathbf{h}) = f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot \mathbf{h} + q(\mathbf{h}) + R_2(\mathbf{h})$$

where

$$\lim_{\mathbf{h} \to 0} \frac{|R_2(\mathbf{h})|}{|\mathbf{h}|^2} = 0. \tag{1}$$

Rewriting and substituting $\nabla f(\mathbf{a}) = 0$, we have

$$f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) = q(\mathbf{h}) + R_2(\mathbf{h}).$$

Suppose that q is positive-definite — that is, $q(\mathbf{h}) > 0$ for all $\mathbf{h}$. Thus in order to show that $\mathbf{a}$ is a local minimum, it suffices to show that $|R_2(\mathbf{h})| < q(\mathbf{h})$ for $\mathbf{h}$ in some neighborhood of 0.

First note that for any $r \in \mathbf{R}$ and $\mathbf{h} \in \mathbf{R}^n$, by the definition of a quadratic form we have

$$q(r\mathbf{h}) = r^2 q(\mathbf{h}).$$

Since the sphere $S^{n-1} = \{\mathbf{x} \in \mathbf{R}^n \ : \ |\mathbf{x}| = 1\}$ is compact (it is clearly bounded, and it is the inverse image of $\{1\}$ under the function $g(x_1, \ldots, x_n) = x_1^2 + \cdots + x_n^2$ so it is closed), the function q attains its minimum m on S^{n-1}; since q is positive-definite, $m \neq 0$. Thus for any nonzero $\mathbf{h} \in \mathbf{R}^n$, we have

$$q(\mathbf{h}) = q\left(|\mathbf{h}| \frac{\mathbf{h}}{|\mathbf{h}|}\right) = |\mathbf{h}^2| \cdot q\left(\frac{\mathbf{h}}{|\mathbf{h}|}\right) \geq m|\mathbf{h}^2|$$

since $\mathbf{h}/|\mathbf{h}| \in S^{n-1}$. By (1), we can find an $\epsilon > 0$ such that $R_2(\mathbf{h}) < m|\mathbf{h}|^2$ when $|\mathbf{h}| < \epsilon$. Thus for all $\mathbf{h}$ such that $|\mathbf{h}| < \epsilon$, we have

$$|R_2(\mathbf{h})| < m|\mathbf{h}|^2 \leq q(\mathbf{h})$$

so we are done.

After I've actually graded the set, there may be some notes on this problem.

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