Homework 7, Part A

Most of you did well on the problem set; the median score was a perfect 10/10 and the mean was a 9.6/10.

2. Suppose $o(f,a) = 0$. This means that $\lim_{\delta \to 0} M(f,a,\delta) - m(f,a,\delta) = 0$. Let ϵ be an arbitrary number greater than zero. Therefore, there exists some δ such that $M(f,a,\delta) - m(f,a,\delta) < \epsilon$. If x is in the open ball centered at a of radius δ, then $|f(x)-f(a)| \leq \max(M(f,a,\delta)-f(a), f(a)-m(f,a,\delta))$, as $m(f,a,\delta) \leq f(x) \leq M(f,a,\delta)$, which is less than or equal to $M(f,a,\delta)-m(f,a,\delta) < \epsilon$; therefore, $|f(x)-f(a)| < \epsilon$ whenever $|x-a| < \delta$; as ϵ is arbitrary, f is continuous at a.

   NOTE: Quite a few people mistakenly stated the definition of continuity at a as "$\exists \delta > 0$ s.t. if $|x-a| < \delta$, then $|f(x)-f(a)| < \epsilon, \forall \epsilon > 0$". Literally, to be less than $\epsilon \forall \epsilon > 0$ means to be less than ANY positive number (and therefore to be either zero or negative); although the mistaken definition looks very similar to the correct definition of "$\forall \epsilon > 0, \exists \delta > 0$ s.t. if $|x-a| < \delta$, then $|f(x)-f(a)| < \epsilon$". For such a mistake I took off half a point if it was obvious that the mistake was merely notational. That's the way the language of mathematics works; the smallest subtleties can lead to huge differences in meaning!

3. Suppose P and P' are two partitions of A. For each positive integer k with $1 \leq k \leq n$, define the finite point set $S_k \subset R$ as follows: a real number x is in S_k iff it is the kth coordinate of some partition point of P or P'. Let $S = S_1 \times S_2 \times \ldots \times S_n$ and let $P*$ be the partition of A such that the partition points of P are precisely the points in S. (We can do this as S, being a finite Cartesian product of finite sets, is finite.) It follows that $P*$ is a refinement of both P and P' by construction; therefore, $L(f,P) \leq L(f,P*) \leq U(f,P*) \leq U(f,P')$ so $L(f,P) \leq U(f,P')$.

   NOTE: This problem was actually done in Page 432 of Fitzpatrick as "Proposition 18.3". Therefore, I counted any solution with at least as much rigor as the text's solution as correct (and allowed usage of Lemma 18.2, the Refinement Lemma, as it was part of the week's readings). As nearly everyone got this problem completely right, I am proud that all of you know how to read a textbook!

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