

# Solution Set 7A

Math 23a  
November 20, 2002

2. Fix an  $n \geq 0$ , and consider the vector space  $P_n$  of polynomials with degree at most  $n$ . If  $0 \leq k \leq n$  then  $P_k \subset P_n$  is a subspace with basis  $\{1, x, x^2, \dots, x^k\}$  and therefore has dimension  $k + 1$ . Since  $D(x^i) = ix^{i-1}$ , the map  $D$  takes basis elements of  $P_k$  into  $P_k$ , and therefore  $P_k$  is invariant under  $D$ . Thus  $P_0, P_1, \dots, P_{n-1}$  are nontrivial invariant subspaces of  $P_n$ ; since  $\dim P_n = n + 1$ , the only possible dimensions of nontrivial subspaces of  $P_n$  are  $1, 2, \dots, n$ , so we have found one in each possible dimension.
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3. a) First we show that  $P_n^0$  and  $P_n^1$  are not invariant under  $D$ , assuming that  $n > 1$  (for, if  $n = 1$  then  $P_n^0 = \{\text{constants}\}$  and if  $n = 0$  then  $P_n^1 = \{0\}$ , so in either case, one is trivially invariant under  $D$ ). We proceed by finding counterexamples: indeed,  $p(x) = x \in P_n^1$  but  $Dp(x) = 1$  has  $Dp(-1) \neq -Dp(1)$ , so  $Dp \notin P_n^1$ . Similarly,  $p(x) = x^2 \in P_n^0$  but  $Dp(x) = 2x$  has  $Dp(1) \neq Dp(-1)$ , so  $Dp \notin P_n^0$ .

For any  $p \in P_n$ , the chain rule gives  $D(p(-x)) = -(Dp)(-x)$ . Therefore if  $p_0 \in P_n^0$  then  $(Dp_0)(x) = D(p_0(-x)) = -(Dp_0)(x)$ , so  $Dp_0 \in P_n^1$ ; similarly, if  $p_1 \in P_n^1$  then  $-(Dp_1)(x) = D(p_1(-x)) = -(Dp_1)(x)$ , so  $Dp_1 \in P_n^0$ . We have thus shown that  $D$  takes  $P_n^0 \rightarrow P_n^1$  and  $P_n^1 \rightarrow P_n^0$ , so in particular,  $P_n^0$  and  $P_n^1$  are invariant under  $D^2$ .

- b) We would like to show that  $P = P_n^0 \oplus P_n^1$ , i.e. that  $P = P_n^0 + P_n^1$  and  $P_n^0 \cap P_n^1 = \{0\}$ .

- We note that if  $p \in P_n^0 \cap P_n^1$  then for all  $x$ , we have  $p(-x) = p(x) = -p(x)$  so  $2p(x) = 0$ , i.e.  $p = 0$ . Thus  $P_n^0 \cap P_n^1 = \{0\}$ .
- If  $p \in P_n$  then we can write

$$p(x) = \frac{p(x) + p(-x)}{2} + \frac{p(x) - p(-x)}{2}.$$

By inspection, we see that the first term is even and the second term is odd, so we have shown that  $p \in P_n^0 + P_n^1$ . Since  $p$  was arbitrary, we have  $P_n = P_n^0 + P_n^1$ .

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Notes on these problems:

- (1) If you did this problem by using bases (despite my constantly harrying you not to use bases if at all possible), then you had to prove that what you used are indeed bases. It says so explicitly in the problem. You cannot just state that  $P_n^0 = \text{span}\{0, x^2, x^4, \dots, x^{\lfloor n/2 \rfloor}\}$  — if you for some reason proved this on a previous problem set, you have to tell me so (because the previous problem set could certainly have been done without showing

this fact). I think that many people were using the definition of  $P_n^0$  from problem set 6 — that is a *different* definition, and it is nontrivial to prove that it is an equivalent one. Please read over the statement of the problem more carefully in the future if this is what you did.

A side note: almost everybody who did attempt to prove that what they had is a basis made at least a small error. Remember, in order to prove that  $P_n^0 = S = \text{span}\{0, x^2, x^4, \dots, x^{\lfloor n/2 \rfloor}\}$  one must prove that (a)  $P_n^0 \subset S$  and (b)  $P_n^0 \supset S$ . The second one is trivial (but you should mention it anyway), and the first one is slightly nontrivial.

- (2) In order to avoid notational hassle in problem 3 (a very understandable thing to do), many people said something like “assume *without loss of generality* that  $n$  is even.” You can’t do this. If you assume that  $n$  is even, then you haven’t proven it for  $n$  odd, so you lost generality. What you should have said is, “the case when  $n$  is odd has a similar proof.” When you say “without loss of generality,  $X$ ,” that means that for some reason, you know that your statement is true when not  $X$  — either you can deduce it from the case when  $X$  is true, or it’s trivial, etcetera. It does *not* just mean that the proof is similar in the not  $X$  case.
- (3) Lots of people said that  $P_0$  is a trivial subspace of  $P_n$ . It’s not. It consists of the scalars, and is one-dimensional, naturally isomorphic to  $\mathbf{R}$ . Recall that in general,  $\dim P_k = k + 1$ .
- (4) One can prove that, in fact,  $P_0, \dots, P_{n-1}$  are the *only* nontrivial subspaces of  $P_n$  that are invariant under  $D$ . How would you go about doing this?

Hint: it is *not* the case that any subspace of  $P_n$  looks like

$$V = \{a_0 + a_1x + \dots + a_nx^n \mid \text{some of the } x_i = 0\}.$$

(Consider, for instance,  $\text{span}(1 + x)$ .) Rather, note that if  $p \in V$  has maximal degree  $k$  then  $Dp \in V$  has degree  $k - 1$ , so you can get rid of the  $x^{k-1}$  term in  $p$ . Use induction on  $k$ .