MATH 23A SOLUTION SET #7 (PART C)

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Problem (5). *Let P_3 be the vector space of polynomials of degree less than or equal to 3, with real coefficients. Let $\mathfrak{B}_1 = \{1, x, x^2, x^3\}$, $\mathfrak{B}_2 = \{1, 1+x, 1+x^2, 1+x^3\}$, and $\mathfrak{B}_3 = \{1+x, 1-x, x^2-x^3, x^2+x^3\}$ be bases for P_3. Let $D : P_3 \to P_3$ be the usual differential operator and let $I : P_3 \to P_3$ be the identity.*

For each of the three bases, write down the matrix for D with respect to that basis (with the basis in question considered as the basis for both domain and range). Also, write down the basis for I where the domain has basis $\mathfrak{B}_1$ and the range has basis $\mathfrak{B}_2$.

Solution. The only thing you need to know to solve this problem is that the columns of the matrix of a linear transformation should be images of the domain basis vectors expressed as linear combinations of the range basis vectors. So here is how it will look in each of the four cases:

(1) Let $e_1 = 1$, $e_2 = x$, $e_3 = x^2$, and $e_4 = x^3$. This is the basis for both the domain and the range.

$$D(e_1) = D(1) = 0 = 0e_1 + 0e_2 + 0e_3 + 0e_4$$

$$D(e_2) = D(x) = 1 = 1e_1 + 0e_2 + 0e_3 + 0e_4$$

$$D(e_3) = D(x^2) = 2x = 0e_1 + 2e_2 + 0e_3 + 0e_4$$

$$D(e_4) = D(x^3) = 3x^2 = 0e_1 + 0e_2 + 3e_3 + 0e_4$$

So the matrix for D with respect to $\mathfrak{B}_1 = \{e_1, e_2, e_3, e_4\}$ is:

$$[D]_{\mathfrak{B}_1,\mathfrak{B}_1} = \begin{pmatrix} | & | & | & | \\ | & | & | & | \\ [D(e_1)]_{\mathfrak{B}_1} & [D(e_2)]_{\mathfrak{B}_1} & [D(e_3)]_{\mathfrak{B}_1} & [D(e_4)]_{\mathfrak{B}_1} \\ | & | & | & | \\ | & | & | & | \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(2) Now let $v_1 = 1$, $v_2 = 1+x$, $v_3 = 1+x^2$, $v_4 = 1+x^3$.

$$D(v_1) = D(1) = 0 = 0v_1 + 0v_2 + 0v_3 + 0v_4$$

$$D(v_2) = D(1+x) = 1 = 1v_1 + 0v_2 + 0v_3 + 0v_4$$

$$D(v_3) = D(1+x^2) = 2x = -2 + 2(1+x) = -2v_1 + 2v_2 + 0v_3 + 0v_4$$

$$D(v_4) = D(1+x^3) = 3x^2 = -3 + 3(1+x^2) = -3v_1 + 0v_2 + 3v_3 + 0v_4$$

So the matrix for D with respect to $\mathfrak{B}_2 = \{v_1, v_2, v_3, v_4\}$ is:

$$[D]_{\mathfrak{B}_2,\mathfrak{B}_2} = \begin{pmatrix} | & | & | & | \\ | & | & | & | \\ [D(v_1)]_{\mathfrak{B}_2} & [D(v_2)]_{\mathfrak{B}_2} & [D(v_3)]_{\mathfrak{B}_2} & [D(v_4)]_{\mathfrak{B}_2} \\ | & | & | & | \\ | & | & | & | \end{pmatrix} = \begin{pmatrix} 0 & 1 & -2 & -3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(3) Now let $w_1 = 1 + x$, $w_2 = 1 - x$, $w_3 = x^2 - x^3$, $w_4 = x^2 + x^3$.

$$D(w_1) = D(1+x) = 1 = \frac{1}{2}(1+x) + \frac{1}{2}(1-x) = \frac{1}{2}w_1 + \frac{1}{2}w_2 + 0w_3 + 0w_4$$

$$D(w_2) = D(1-x) = -1 = -\frac{1}{2}(1+x) - \frac{1}{2}(1-x) = -\frac{1}{2}w_1 - \frac{1}{2}w_2 + 0w_3 + 0w_4$$

$$D(w_3) = D(x^2-x^3) = 2x-3x^2 = (1+x)-(1-x)-\frac{3}{2}(x^2-x^3)-\frac{3}{2}(x^2+x^3) = 1w_1-1w_2-\frac{3}{2}w_3-\frac{3}{2}w_4$$

$$D(w_4) = D(x^2+x^3) = 2x+3x^2 = (1+x)-(1-x)+\frac{3}{2}(x^2-x^3)+\frac{3}{2}(x^2+x^3) = 1w_1-1w_2+\frac{3}{2}w_3+\frac{3}{2}w_4$$

So the matrix for D with respect to $\mathfrak{B}_3 = \{w_1, w_2, w_3, w_4\}$ is:

$$[D]_{\mathfrak{B}_3,\mathfrak{B}_3} = \begin{pmatrix} | & | & | & | \\ | & | & | & | \\ [D(w_1)]_{\mathfrak{B}_3} & [D(w_2)]_{\mathfrak{B}_3} & [D(w_3)]_{\mathfrak{B}_3} & [D(w_4)]_{\mathfrak{B}_3} \\ | & | & | & | \\ | & | & | & | \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 & 1 & 1 \\ 1/2 & -1/2 & -1 & -1 \\ 0 & 0 & -3/2 & 3/2 \\ 0 & 0 & -3/2 & 3/2 \end{pmatrix}$$

(4) We use the same procedure as in the previous to cases to find $[I]_{\mathfrak{B}_1,\mathfrak{B}_2}$:

$$I(e_1) = I(1) = 1 = 1v_1 + 0v_2 + 0v_3 + 0v_4$$

$$I(e_2) = I(x) = x = -1 + (1+x) = -1v_1 + 1v_2 + 0v_3 + 0v_4$$

$$I(e_3) = I(x^2) = x^2 = -1 + (1+x^2) = -1v_1 + 0v_2 + 1v_3 + 0v_4$$

$$I(e_4) = I(x^3) = x^3 = -1 + (1+x^3) = -1v_1 + 0v_2 + 0v_3 + 1v_4$$

So the matrix for I with respect to $\mathfrak{B}_1$ as a basis for the domain and $\mathfrak{B}_2$ as a basis for the range is:

$$[I]_{\mathfrak{B}_1,\mathfrak{B}_2} = \begin{pmatrix} | & | & | & | \\ | & | & | & | \\ [I(e_1)]_{\mathfrak{B}_2} & [I(e_2)]_{\mathfrak{B}_2} & [I(e_3)]_{\mathfrak{B}_2} & [I(e_4)]_{\mathfrak{B}_2} \\ | & | & | & | \\ | & | & | & | \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\square$