MATH 23b, SPRING 2004
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Homework Assignment # 8
Due: April 16, 2004

Homework Assignment #8 (Final Version)

1. Read Edwards, Sections 4.1–4.3.

2. (A) Use the Spectral Theorem to diagonalize the quadratic form
   $q(x, y, z) = xy + yz + xz$ and to classify its "definite"-ness.

3. (B) Let $A \subset \mathbb{R}^n$ be a closed rectangle, and let $f : A \longrightarrow \mathbb{R}$ be a bounded
   function. If P and P' are any two partitions of A, show that

   $$L(f, P) \leq U(f, P').$$

4. (C) Let $f : A \longrightarrow \mathbb{R}$ be integrable. Show that $|f|$ is integrable and that

   $$\left| \int_A f \right| \leq \int_A |f|.$$

5. (*) Let $A \subset \mathbb{R}^n$ be a closed rectangle, and let

   $$I = \{f : A \longrightarrow \mathbb{R} \mid f \text{ is integrable on } A\}.$$

   (a) Show that I is a vector space over $\mathbb{R}$ by showing that if $f_1, f_2 \in I$
       and $c_1, c_2 \in \mathbb{R}$, then $c_1 f_1 + c_2 f_2 \in I$. (In other words, I is a subspace
       of the vector space $V = \{f : A \longrightarrow \mathbb{R}\}$.)
   (b) Show that $\int_A cf = c \int_A f$ and $\int_A (f_1 + f_2) = \int_A f_1 + \int_A f_2$.

6. (D) Let $A = [0, 1] \times [0, 1]$, and define $f : A \longrightarrow \mathbb{R}$ as follows:

   $$f(x, y) = \begin{cases} 0 & , \quad \text{if } x \notin \mathbb{Q} \\ 0 & , \quad \text{if } x \in \mathbb{Q} \text{ and } y \notin \mathbb{Q} \\ \frac{1}{q} & , \quad \text{if } x \in \mathbb{Q} \text{ and } y = \frac{p}{q} \text{ in lowest terms} \end{cases}$$

   (a) For each $\mathbf{a} \in A$, determine $o(f, \mathbf{a})$.
   (b) Show that f is integrable on A. (What is $\int_A f$?)

7. (deferred) Give an example of a closed set of measure zero that does not
   have content zero.

8. (deferred) Let $A = \{x \in [0, 1] \mid$ the decimal expansion of x has no 8's$\}$.

Let $B = \{n \in \mathbb{N} \mid$ the decimal expansion of n has no 8's$\}$.

(a) Find the content of A.

(b) Decide whether the infinite series $\displaystyle\sum_{n \in B} \frac{1}{n}$ converges or diverges.

9. (deferred) For a function $f : [0, 1] \longrightarrow \mathbb{R}$, let $A = \{x \in [0, 1] \mid f$ is not differentiable at $x\}$. Find such an f satisfying the following conditions:

- f is continuous.
- $f(0) = 0$
- $f(1) = 1$
- A has content zero.
- If $x \notin A$, then $f'(x) = 0$.

(Hint: Use the Cantor set.)