

MATH 23b, SPRING 2004
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Homework Assignment # 9
Due: April 23, 2004

Homework Assignment #9 (Final Version)

1. Read Edwards, Sections 4.1–4.3.
2. (A) In class, we proved the following theorem:

Theorem. Let $A \subset \mathbb{R}^n$ be a closed rectangle, and let $f : A \rightarrow \mathbb{R}$ be a bounded function. If f is continuous at a , then $o(f, a) = 0$.

Prove the converse. (Recall that $o(f, a)$ is the *oscillation* of f at a .)

3. (*) Give an example of a closed set of measure zero that does not have content zero.
4. (A) Let $A = \{x \in [0, 1] \mid \text{the decimal expansion of } x \text{ has no 8's}\}$.
Let $B = \{n \in \mathbb{N} \mid \text{the decimal expansion of } n \text{ has no 8's}\}$.

(a) Find the content of A .

(b) Decide whether the infinite series $\sum_{n \in B} \frac{1}{n}$ converges or diverges.

(Hint: These two parts actually have very little to do with each other, except superficially.)

5. (*) Find a bijection between the Cantor set and the closed interval $[0, 1]$.
(Hint: Consider the ternary expansion of numbers in the Cantor set.)
6. (B) For a function $f : [0, 1] \rightarrow \mathbb{R}$, let $A = \{x \in [0, 1] \mid f \text{ is not differentiable at } x\}$. Find such an f satisfying the following conditions:
 - f is continuous.
 - $f(0) = 0$
 - $f(1) = 1$
 - A has content zero.
 - If $x \notin A$, then $f'(x) = 0$.

(Hint: Use the Cantor set.)

7. (C) In class, we proved the following theorem:

Theorem. Let $A \subset \mathbb{R}^n$ be a closed rectangle, and let $f : A \rightarrow \mathbb{R}$. Let $B = \{x \in A \mid f \text{ is discontinuous at } x\}$. Then f is integrable if and only if B has measure zero.

In the “if” part of the proof, we considered the set:

$$B^\varepsilon = \{x \in A \mid o(f, x) \geq \varepsilon\}$$

and claimed that it was closed (and hence compact since A is bounded). Prove this.

8. (D) Let $f : A \rightarrow \mathbb{R}$ and $g : A \rightarrow \mathbb{R}$ be two bounded functions such that the closure of $B = \{x \in A \mid f(x) \neq g(x)\}$ is a set of measure zero. (This implies that f and g agree except on a set of measure zero.) Show that f is integrable if and only if g is, and that if they are both integrable, then $\int_A f = \int_A g$.

9. (deferred) In class, we considered the set $A = \mathbb{Q} \cap [0, 1]$ and showed that it has measure zero. In particular, we showed that it was countable, that is, we could write it as $A = \{a_0, a_1, a_2, a_3 \dots\}$. Given an $\varepsilon > 0$, we then covered it with rectangles $I_i = [a_i - \frac{\varepsilon}{2^{i+2}}, a_i + \frac{\varepsilon}{2^{i+2}}], \forall i \in \mathbb{N}$, so that

$$v\left(\bigcup_{i=0}^{\infty} I_i\right) \leq \sum_{i=0}^{\infty} v(I_i) = \varepsilon.$$

Fix $\varepsilon = \frac{1}{2}$, and let $J_i = (a_i - \frac{1}{2^{i+3}}, a_i + \frac{1}{2^{i+3}})$ be the open rectangle equal to the interior of the corresponding I_i defined above.

Let $B = \bigcup_{i=2}^{\infty} J_i$. (Note that we are purposely omitting the first two sets, which cover 0 and 1, respectively, so that each $J_i \subset [0, 1]$!)

- Show that $\partial B = [0, 1] \setminus B$.
- Show that ∂B does not have measure zero.
- Let χ_B be the characteristic function of B . Show that χ_B is not integrable on $[0, 1]$.

(Note that although B is a “reasonable” set in the sense that it is the union of a countable collection of open sets, it does not have a “reasonable” boundary, and so χ_B is not integrable.)