MATH 23a, FALL 2001
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Lecture # 23, supplement

Alternating n-Linear Forms Applied to Linearly Independent Vectors

Theorem: Let V be a vector space over F of dimension n. Let $f : V^n \longrightarrow F$ be a non-zero alternating multilinear form. If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a collection of linearly independent vectors in V, then $f(\mathbf{v}_1, \ldots, \mathbf{v}_n) \neq 0$.

Remark: It is important to note that this theorem is true only for alternating n-linear forms when $dim(V) = n$. See the homework exercise (HW # 7.4) which shows that this statement is false for an alternating k-linear form when $k < n$.

Proof: For simplicity of notation but without any loss of generality, we give the proof for $n = 3$. Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be the three linearly independent vectors.

Let $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \in V$ be any three vectors. Since the $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ form a basis, we may write each of the $\mathbf{w}_i$ as a linear combination of the $\mathbf{v}_i$:

$$\begin{aligned}
\mathbf{w_1} &= \alpha_{11}\mathbf{v}_1 + \alpha_{21}\mathbf{v}_2 + \alpha_{31}\mathbf{v}_3 \\
\mathbf{w_2} &= \alpha_{12}\mathbf{v}_1 + \alpha_{22}\mathbf{v}_2 + \alpha_{32}\mathbf{v}_3 \\
\mathbf{w_3} &= \alpha_{13}\mathbf{v}_1 + \alpha_{23}\mathbf{v}_2 + \alpha_{33}\mathbf{v}_3
\end{aligned}$$

Hence

$$\begin{aligned}
f(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) &= f(\alpha_{11}\mathbf{v}_1 + \alpha_{21}\mathbf{v}_2 + \alpha_{31}\mathbf{v}_3, \alpha_{12}\mathbf{v}_1 + \alpha_{22}\mathbf{v}_2 + \alpha_{32}\mathbf{v}_3, \alpha_{13}\mathbf{v}_1 + \alpha_{23}\mathbf{v}_2 + \alpha_{33} \\
&= \alpha_{11} f(\mathbf{v}_1,\ \alpha_{12}\mathbf{v}_1 + \alpha_{22}\mathbf{v}_2 + \alpha_{32}\mathbf{v}_3,\ \alpha_{13}\mathbf{v}_1 + \alpha_{23}\mathbf{v}_2 + \alpha_{33}\mathbf{v}_3) \\
&\quad + \alpha_{21} f(\mathbf{v}_2,\ \alpha_{12}\mathbf{v}_1 + \alpha_{22}\mathbf{v}_2 + \alpha_{32}\mathbf{v}_3,\ \alpha_{13}\mathbf{v}_1 + \alpha_{23}\mathbf{v}_2 + \alpha_{33}\mathbf{v}_3) \\
&\quad + \alpha_{31} f(\mathbf{v}_3,\ \alpha_{12}\mathbf{v}_1 + \alpha_{22}\mathbf{v}_2 + \alpha_{32}\mathbf{v}_3,\ \alpha_{13}\mathbf{v}_1 + \alpha_{23}\mathbf{v}_2 + \alpha_{33}\mathbf{v}_3) \\
&= \sum_{i=1}^{3} \alpha_{i1} \cdot f(\mathbf{v}_i,\ \alpha_{12}\mathbf{v}_1 + \alpha_{22}\mathbf{v}_2 + \alpha_{32}\mathbf{v}_3,\ \alpha_{13}\mathbf{v}_1 + \alpha_{23}\mathbf{v}_2 + \alpha_{33}\mathbf{v}_3) \\
&\qquad \text{after expanding by multilinearity in the first component} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} \alpha_{i1}\alpha_{j2} \cdot f(\mathbf{v}_i,\ \mathbf{v}_j,\ \alpha_{13}\mathbf{v}_1 + \alpha_{23}\mathbf{v}_2 + \alpha_{33}\mathbf{v}_3) \\
&\qquad \text{after expanding by multilinearity in the second component} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \alpha_{i1}\alpha_{j2}\alpha_{k3} \cdot f(\mathbf{v}_i,\ \mathbf{v}_j,\ \mathbf{v}_k) \\
&\qquad \text{after expanding by multilinearity in the third component}
\end{aligned}$$

Now we use the assumption that f is alternating to observe that any term in the sum with two or more of the $\mathbf{v}$'s being equal must be zero. In fact, we can reduce this sum to one where $i \neq j$, $i \neq k$, and $j \neq k$, in other words, where (i, j, k) is a permutation of $(1, 2, 3)$.

Thus,

$$\begin{aligned} f(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) &= \sum_{\pi \in S_3} \alpha_{\pi(1)1} \alpha_{\pi(2)2} \alpha_{\pi(3)3} \cdot f(\mathbf{v}_{\pi(1)}, \mathbf{v}_{\pi(2)}, \mathbf{v}_{\pi(3)}) \\ &= \sum_{\pi \in S_3} \operatorname{sgn}(\pi) \alpha_{\pi(1)1} \alpha_{\pi(2)2} \alpha_{\pi(3)3} \cdot f(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \end{aligned}$$

Finally, we argue by contradiction. Note that this last expression depends on the particular set of linearly independent vectors $\mathbf{v}$ given in the statement of the theorem. Suppose $f(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = 0$. Then the argument presented above shows that $f(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) = 0$ for *any* vectors in V since the $\mathbf{w}$'s were specified only in terms of the coefficients α. But this implies that f was identically zero, that is, the 0 multilinear form, which contradicts our assumptions.

The proof above was given in the $n = 3$ case. Note that there were $27 = 3^3$ terms in the longest expansion, but that only $6 = 3!$ of these were non-zero. In general, there would be n^n terms, only $n!$ of which are non-zero. We hope that this simpler version of the proof clarifies the general one.